Allocation of Periodic Task Modules with Precedence and Deadline Constraints in Distributed Real-Time Systems

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Abstract—This paper addresses the problem of allocating (assigning and scheduling) periodic task modules to processing nodes in distributed real-time systems subject to task precedence and timing constraints. Using the branch-and-bound technique, a module allocation scheme is proposed to find an “optimal” allocation that maximizes the probability of meeting task deadlines.

The task system within a planning cycle is first modeled with a task flow graph which describes computation and communication modules, as well as the precedence constraints among them. To incorporate both timing and logical correctness into module allocation, the probability of meeting task deadlines is used as the objective function. The module allocation scheme is then applied to find an optimal allocation of task modules in a distributed system. The timing aspects embedded in the objective function drive the scheme not only to assign task modules to processing nodes, but also to use a module scheduling algorithm (with polynomial time complexity) for scheduling all modules assigned to each node, so that all tasks may be completed in time.

In order to speed up the branch-and-bound process and to reduce the computational complexity, a dominance relation is derived from the requirement of timely completion of tasks and use to eliminate the possibility of generating vertices in the state-space search tree, which never lead to an optimal solution, and an upper bound of the objective function is derived for every partial allocation with which the scheme determines whether or not to prune the corresponding intermediate vertex in the search tree. Several numerical examples are presented to demonstrate the effectiveness and practicality of the proposed scheme.

Index Terms—Real-time systems, dynamic failure, task/module allocation, module scheduling, precedence and deadline constraints, task flow graph, branch-and-bound process.

1 INTRODUCTION

The availability of inexpensive, high-performance processors and high-capacity memory chips has made it attractive to use distributed computing systems for real-time applications. For example, one can make the execution of both periodic and aperiodic tasks not only logically correct, but also completed before their deadlines, by partitioning periodic tasks into a set of communicating modules, statically allocating these modules to processing nodes (PNs) in a distributed system, and dynamically distributing aperiodic tasks as they arrive according to the load state of each PN.

Partitioning tasks are usually based on some application-dependent criterion and the system architecture under consideration, while the dynamic distribution of aperiodic tasks is usually treated as an adaptive load sharing problem. Both of these are not the intent of this paper; see [1] for an example of partitioning real-time tasks into modules/activities, and see [2], [3], [4], [5], [6] for examples of dynamic load sharing in distributed real-time systems. In this paper, we consider, instead, the issue of “optimally” (in the sense to be defined later) allocating periodic task modules to PNs in a distributed system, so as to fully utilize the inherent parallelism, modularity, and reliability of the system, while alleviating the “saturation effect” [7] caused by excessive interprocessor communication of data and control messages.

The problem of allocating tasks/modules in a distributed system has been studied by many researchers with respect to different objective functions. These objective functions can be roughly grouped into four categories:

1) Minimization of total computation and communication times in the system [7], [8], [9], [10]. In the case of homogeneous systems, this objective function reduces to the minimization of the total interprocessor communication time.

2) Load balancing by minimizing the statistical variance of processor utilization [11], [12] or by maximizing the total rewards in the semi-Markov process that models the computer system [13].

3) Minimization of maximum computation and communication times on a PN, the objective function of which was termed the maximum turnaround time in [14], the bottleneck processor time in [15], [16], and the system hazard in [17].

4) Maximization of the reliability function of both PNs and communication links [18], [19].

Different objective functions lead to different optimality conditions and different allocation results. The first two
objectives are suitable for a distributed system executing multiple simultaneous non-real-time applications, where maximizing the total throughput or minimizing the average response time is the main concern. However, for real-time systems, the timing and logical correctness of each individual task must be considered, because failure to correctly complete a task in time could cause catastrophe. Thus, the third objective function, which is based on the worst-case behavior, is more suitable for assessing the timeliness of real-time systems, while the fourth objective function, that incorporates reliability into task/module allocation, is more suitable for assessing logical correctness. In this paper, we use the probability of completing each task with both timing and logical correctness as the objective function, which was termed, in [20], the probability of no dynamic failure \( P_{ND} \). Specifically, \( P_{ND} \) is the product of two component probabilities:

- The probability, \( P_{NDL} \), that all tasks within a planning cycle are completed before their deadline. The planning cycle is the time period within which the task-invocation behavior repeats itself throughout the entire mission, and, thus, completely specifies the entire task system. More on this will be discussed in Section 2.
- The probability, \( P_{NDR} \), that all PNs are operational during the execution of task modules assigned to them, and the links between communicating PNs are operational for all intermodule communications over these links under a given allocation.

Consequently, the use of \( P_{ND} \) incorporates both timeliness and logical correctness into task allocation. This is in sharp contrast to the other module allocation approaches reported in the literature, which deal with either average task response time or logical correctness, but not both.

The problem of finding an optimal assignment of tasks/modules to processors subject to precedence constraints has been shown to be NP-hard for all the above problem formulations [21], [22], and some form of enumerative optimization and/or local search approaches must be sought. In this paper, we develop a module allocation (MA) scheme using the branch-and-bound (BB) method. We first model the task system with a task flow graph (TG), which describes computation and communication modules as well as the precedence constraints among them. We then use the BB method to search for an optimal module allocation. The computational complexity is reduced by deriving an upper bound of the objective function with which we determine whether to expand or prune intermediate vertices (corresponding to partial allocations) in the state-space search tree. On the other hand, because of the timing aspects embedded in the objective function, the performance of any resulting assignment strongly depends on how the assigned tasks/modules are scheduled. Thus, when evaluating an upper-bound (exact) objective function for a partial (complete) allocation, we use a module scheduling (MS) algorithm (with polynomial time complexity) to schedule all the modules assigned to a PN by minimizing the maximum tardiness of modules subject to precedence constraints. The MA scheme, combined with the MS algorithm, is guaranteed to find the optimal allocation of modules to PNs subject to precedence constraints. By “allocation,” we mean the assignment of modules coupled with the scheduling of all modules assigned to each PN.

Shen and Tsai [14] minimized the maximum turnaround time, but considered only a single invocation of each task. They did not take into account precedence constraints between tasks. Chu et al. [15], [16] chose to minimize the bottleneck processor workload for tasks/modules assignment, but their algorithm/analysis was solely based on mean task response times, which eliminates the need to consider the scheduling problem. Peng and Shin [17] are the first to include the important timing aspects in the objective function, and combine task scheduling with task assignment. They chose to minimize the system hazard, which is defined as the maximum normalized (with respect to task period) task flowtime. It is, however, not clear how system hazard is related to the probability of no dynamic failure. The restriction on assigning all modules of the same task to a single PN may not always be desirable.

Ramanritham [23] used a heuristic-directed search technique with tunable design parameters to

1. By communicating PNs, we mean a pair of PNs to which two communicating modules are assigned under a given allocation.
2 TASK AND SYSTEM MODELS

2.1 The Task System

Real-time tasks are either periodic or nonperiodic. A periodic task is invoked at fixed time intervals and constitutes the base load of the system. Its attributes, such as the required resources, the execution time, and the invocation period, are usually known a priori. A nonperiodic task, on the other hand, is invoked randomly in response to environmental stimuli, especially to unanticipated abnormal situations. The main intent of this paper is to address the problem of allocating the modules of periodic tasks.

2.1.1 Planning Cycle

To analyze the behavior of periodic tasks, we only need to consider the task behaviors within a specific period, the task behaviors during which will repeat for the entire mission lifetime. Such a period is called the planning cycle of periodic tasks and is defined as the least common multiple (LCM) \( L = \{ p_i : i = 1, 2, ..., N_T \} \), where \( p_i \) is the period of a task \( T_i \) and \( N_T \) is the total number of periodic tasks in the system. That is, the planning cycle is the time interval \( t_0 + kL \), \( t_0 + (k + 1)L \) where \( t_0 \) is the mission start time, and \( k \) is a nonnegative integer.

2.1.2 Attributes and Precedence Constraints Among Modules

Each task can be decomposed into smaller units, called modules [1]. Each module \( M_i \) requires \( e_i \) units of execution time. The execution time of a module could be its worst-case execution time or its real execution time, if known. Since extensive simulations and testing are required before putting any critical real-time system in operation (e.g., fly-by-wire computers), the system designer is assumed to have a good, albeit sometimes incomplete, understanding of either the exact execution time or the worst-case execution time of each module.

The execution order of modules imposes precedence constraints among them. These precedence constraints are of the form \( M_i \rightarrow M_o \), meaning that the completion of \( M_i \) of a task enables another module \( M_o \) of the same task to be ready for execution (e.g., by letting \( M_i \) send a short message to enable \( M_o \)'s execution and/or update the data variables/files shared between them [15], [16]). On the other hand, tasks communicate with one another to accomplish the overall control mission. The semantics of message communication between two cooperating tasks also impose precedence constraints between the associated modules of these tasks. This kind of precedence constraint is also of the form \( M_i \rightarrow M_o \), except that \( M_i \) and \( M_o \) now belong to different tasks.

If \( M_i \) and \( M_o \) are assigned to the same PN, communication between them can be achieved via accessing shared memory. Overheads of such communications are usually much smaller than those when \( M_i \) and \( M_o \) reside on different PNs. Any two communicating modules that reside on two different PNs will incur interprocessor communication (IPC). IPC introduces a communication delay, which is a function of intermodule communication (IMC) volume (measured in data units) and the worst case link delay (or delay per data unit) between the two communicating PNs.

It is important to observe that, even if exact module execution times were known in advance, task execution times are not known, due to, for instance, the existence of data-dependent (thus, probabilistic) branches/loops in the task (to be discussed below) and/or inexact knowledge of IMC/IPC delays.

2.1.3 Task Flow Graph (TG)

A TG is commonly used to describe the logical structure of modules, and the communications and precedence constraints among them. A TG is composed of four types of subgraphs: chain, AND-subgraph, OR-subgraph, and loop. See [1], [24] for a detailed account of the four component subgraphs. Here we assume that the probability for taking a particular branch in an OR-subgraph or for repeating/exiting the body of a loop is assumed to be independent of that for others (residing either in modules of different tasks or in modules of the same task). These probability values could be set to worst-case values and can be obtained from the extensive simulations and testing—usually required of critical real-time systems—during the system design phase. We also assume that the number of times a loop can be executed is no more than its maximum loop count. Imposing a maximum loop count for each loop is necessary for real-time applications, since each real-time task must be completed in a finite time. Fig. 1a shows a simple example of a TG.

2.1.4 Communication Primitives

The semantics of the most general communication primitive SEND-RECEIVE-REPLY, can be embedded into precedence relations between modules. If module \( M_i \) of task \( T_i \) issues a SEND to task \( T_j \), \( T_i \) remains blocked, or cannot execute module \( M_o \) that follows \( M_i \) until the corresponding REPLY from \( T_j \) is received. If the module, \( M_o \), responsible for the corresponding communication activity on \( T_j \)'s side executes a RECEIVE before the SEND arrives, \( T_j \) also remains blocked. For example, the communication activities between tasks in Fig. 1a can be embedded into the precedence constraints between modules, as shown in Fig. 1b.

2.2 The Distributed System

The distributed system considered here consists of \( K \) processing nodes (PNs). For clarity of exposition, all PNs are assumed to have the same processing power and the same set of resources. (This assumption can, however, be easily relaxed, but with more complex notation.) The time required by an IMC within a PN is assumed to be negligible, while that between two PNs is expressed as the product of the IMC volume (measured in data units) and the worst case link delay (measured in time units per data unit) between the two PNs on which the communicating modules reside.

Note that the worst case link delay is the communication delay bound guaranteed by the underlying communication subsystem to provide to messages with time constraints. Here, we

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2. See Section 2.2 for the reason of expressing IPC as a function of IMC volume and worst case link delay.
3. Other communication primitives, such as QUERY–RESPONSE and WAITFOR [1], can always be realized using SEND–RECEIVE–REPLY.
4. The time for packetization and depacketization is lumped into module execution time for the clarity of algorithm description.
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assume that the communication subsystem and the underlying protocol support time-constrained communications, and the worst-case delay experienced by time-constrained messages is bounded and predictable. Examples of such communication subsystems are the highly responsive token ring described in [25], the FDDI network with the timed-token MAC protocol [26], [27], [28], [29], [30], the distributed queue dual bus (DQDB) network with the isochronous services [31], [32], and the point-to-point packet-switched network described in [33], [34]. No restriction is imposed on the topology of the communication subsystem.

Each PN $N_k$ and each link $\ell_{mn}$ between $N_m$ and $N_n$ are assumed to fail independently with exponential rates $\lambda_k$ and $\lambda_{mn}$, respectively. We do not take into account of repair and recovery times for failed PNs in assessing the logical correctness of an allocation. Instead, we shall allocate modules to PNs on which failures are least likely to occur during the execution of task modules. That is, we shall assign modules to PNs with maximum reliability and thus eliminate the need of on-line repair and recovery. To guard against the unlikely failures of these PNs, one can assign copies of a module to multiple PNs, but this subject is not the scope of this paper. The rationale behind the above assumption is that

1) repair and recovery times are largely implementation-dependent, and
2) repair and recovery routines usually introduce too high time overheads to be used on-line for time-critical applications.

**3 Module Allocation Algorithm**

Let $N$ be the number of modules to be allocated within a planning cycle. The module allocation problem can be formulated as that of maximizing $P_{ND}(x) = P_{ND1}(x) \cdot P_{ND2}(x)$ over all possible allocations subject to

$$\sum_{k=1}^{K} x_{ik} = 1, \quad \text{for } 1 \leq i \leq N,$$

where $x_{ik}$ is the probability that all tasks meet their deadlines under allocation $x$, and $P_{ND}(x)$ is the probability that all modules are operational during the execution of modules assigned to them and all communication links are operational during the IPCs that use these links under $x$. As will be clear later, the precedence constraints among modules are figured in the calculation of module release times (to be defined later), and the timing constraints on modules/tasks are considered when $P_{ND}(x)$ is evaluated; for example, $P_{ND}(x) = 0$ if some task in the TG misses its deadline under $x$. The expressions for $P_{ND}(x)$ and $P_{ND2}(x)$ will be derived in Sections 4 and 5.

To solve the above module allocation problem, we propose the MA scheme which uses:

- **Branch-and-Bound (BB) method** to implicitly enumerate all possible allocations while effectively pruning unnecessary paths in the search tree.
- **Module Scheduling (MS) algorithm** to schedule the modules assigned to each PN subject to precedence constraints and latest module completion times.

**Fig. 1.** An example of task flow graph. The label that appears in the upper right corner of each box in (b) is the acyclic number associated with each module.
The BB method enumerates all possible solutions to a given problem by "growing" the corresponding search tree. Each intermediate (leaf) vertex in the search tree corresponds to a partial (complete) allocation. This method is composed of two procedures: branching and bounding. The branching process generates the child vertices of an intermediate vertex \( x \) in the search tree, until an optimal solution is completely specified. Usually a dominance relation is derived to limit the number of child vertices generated for each intermediate vertex \( x \) without eliminating any path to an optimal solution. On the other hand, the bounding process calculates a tight upper bound of the objective function (UBOF) for each newly-generated vertex \( x \), based on which one can decide whether or not \( x \) may lead to an optimal solution. If the UBOF of a vertex \( x \) is less than the current best objective found in the search process, then \( x \) will never lead to an optimal solution, and should thus be pruned. The interested reader is referred to [35] for a detailed account of the branch-and-bound method.

The MA scheme works as follows: All modules in the task system are assumed to be numbered in acyclic order, such that, if \( M_i \rightarrow M_j \), then \( i < j \). For example, the numbers which appear on the upper right corner of the boxes in Fig. 1b give an example of acyclic numbering. The MA scheme begins with a null allocation \( x_0 \) which corresponds to the root of the search tree, and allocates modules in the order of their acyclic numbering. Let \( TG(x) \) denote the set of modules which are already allocated under \( x_0 \) and AN the set of active vertices in the search tree to be considered for expansion. (AN is determined by the bounding test.)

Expanding a vertex \( x \in AN \) corresponds to allocating the module, \( M_i \), with the smallest acyclic number in \( TG \setminus TG(x) \) to a PN, where \( \setminus \) denotes the difference of two sets. Only those PNs which have enough idle times to ensure the timely completion of \( M_i \) and survive the branching test will be considered as candidates for allocating \( M_i \). After the expanding operation is performed, the bounding test is applied to those vertices expanded from \( x \) by allocating \( M_i \) to one of the candidate PNs. The UBOF, \( \hat{P}_{ND}(x) \), of each newly-generated (intermediate) vertex \( y \) is calculated by scheduling modules \( \in TG(y) \) with MS and evaluating \( P_{ND}(y) \) and \( P_{ND2}(y) \) with the expressions derived in Sections 4 and 5, respectively. If a vertex \( y \) has its \( \hat{P}_{ND}(y) \) greater than the current best objective function value \( P_{ND}^{*} \), it survives the bounding test, might possibly lead to the optimal solution, and will be made active and considered for vertex expansion in the next stage; otherwise, it will be pruned. The scheme terminates when an optimal solution is found.

The MA scheme is outlined below. The MS algorithm which schedules the modules assigned to each PN (under the given allocation and the given timing constraints) will be discussed in Section 4.1. The expressions of \( P_{ND1}(y) \) and \( P_{ND2}(y) \) (used to assess the likelihood of the timeliness and the logical correctness of an allocation \( y \)) will be derived in Sections 4.2 and 5, respectively. The branching and bounding tests used to achieve BB efficiency will be treated in Sections 6.1 and 6.2, respectively.

4 EVALUATION OF TIMELINESS

In this section, we evaluate \( P_{ND}(x) \) for a given allocation \( x \). We first describe how MS schedules all the modules assigned to a PN, say \( N_i \), under \( x \) to minimize the maximum module tardiness subject to task release times and precedence constraints. By applying MS to each PN, we can obtain a module schedule under \( x \). Second, we calculate the probability, \( P(T_{\ell} \text{ is timely completed under } x) \), for every \( T_{\ell} \) in TG. \( P_{ND1}(x) \) can then be calculated from \( P(T_{\ell} \text{ is timely completed under } x) \).

4.1 The Module Scheduling Algorithm

To facilitate the description and analysis of MS, we need to introduce the following notation:

- **TG**: a component task graph of TG. If TG contains loops or OR-subgraphs, it will be replaced by a set of component task graphs without loops and OR-graphs before applying MS (see Section 4.2 for more on this). For the time-being, we only need to know that TG contains neither loops nor OR-subgraphs.

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5. \( TG(x) = TG \) if \( x \) is a complete allocation.
the modules in \( \mathbf{MS} \) are to be scheduled preemptively on \( \mathbf{MS} \). Each module \( M_i \) has a release time \( r_i \); the task system. If \( T_i \) is not scheduled to be executed upon its release, it is used to include the effect of queuing \( M_i \) on the release times of all the modules that succeed \( M_i \). The modified execution time of \( M_i \), \( \hat{\epsilon}_i \), is obtained as follows:

\[
\hat{\epsilon}_i = \begin{cases} 
\epsilon_i & \text{if } M_i \text{ is scheduled to be executed upon its release at time } r_i \\
\epsilon_i - (C_i - r_i) & \text{otherwise.}
\end{cases}
\]

Before proceeding to describe and analyze \( \mathbf{MS} \), we define the cost function \( f(C) \) and discuss how to calculate the two parameters, \( LC_i \) and \( r_i \) \( \forall i \). The cost function is defined as

\[
f(C) = C_i - LC_i
\]  
(4.1)

where \( LC_i \) is the latest time \( M_i \) must be completed to ensure the timeliness of all of its succeeding modules, and \( C_i \) is the completion time of \( M_i \) determined by \( \mathbf{MS} \). If \( C_i > LC_i \), a positive cost will occur. Thus, with the definition of this cost function, minimizing the maximum cost function is equivalent to minimizing the maximum tardiness of modules in \( \mathbf{TG} \).

The latest completion time, \( LC_i \) of \( M_i \in \mathbf{TG} \) is obtained as follows: Let \( LC_i \) be initially set to the deadline of the task to which \( M_i \) belongs. Then, modify \( LC_i \) as

\[
LC_i = \min\left\{ LC_i, \min\{LC_j - \epsilon_j - com_j(x) : M_j \rightarrow M_i\}\right\},
\]

\( i = N - 1, \ldots, 1 \)
(4.2)

where the modules are assumed to be numbered in acyclic order and

\[
com_j(x) = \begin{cases} 
0 & \text{if } M_i \text{ and } M_j \text{ are assigned to the same PN under } x, \\
\min d_{ij} r_{ij} m_{ij} & \text{if } M_i \text{ and } M_j \text{ are assigned to } N_{ij} \text{ and } N_{ij} \text{ under } x, \text{ respectively.}
\end{cases}
\]

Note that (4.2) computes backward from \( i = N - 1 \) to \( i = 1 \), because \( \mathbf{MS} \) has no successor by the nature of acyclic order, and, thus, the latest completion time of \( M_{N_i} \) is exactly the deadline of the task it belongs to. When \( x \) is a partial allocation and either \( M_i \) or \( M_i \) or both have not yet been assigned, \( com_j(x) \) is (optimistically) assumed to be 0.

The release time, \( r_i \), of \( M_i \in \mathbf{TG}(x) \) is obtained as follows: Let \( r_i \) be initially set to the invocation time of the task to which \( M_i \) belongs. Then, modify \( r_i \) as

\[
r_i = \max\left\{ r_j + \hat{\epsilon}_j + com_j(x) : M_j \rightarrow M_i\right\},
\]

\( 2 \leq i \leq N \)
(4.3)

where \( r_i \) is the invocation time of the task to which \( M_i \) belongs, and \( \hat{\epsilon}_j = \max\{C_j - r_j, \epsilon_j\} \) is the modified execution time which equals the sum of \( M_j \)'s execution time, \( \epsilon_p \) and \( M_i \)'s queuing time (if \( M_i \) is not scheduled to be executed upon its release). \( \hat{\epsilon}_j \) is used to include the effect of queuing \( M_j \)'s preceding module, \( M_j \), on \( M_i \)'s release time.

Note that the modified execution times of all \( M_i \)'s preceding modules must be available prior to the calculation of \( r_i \). This is ensured by allocating the modules in the order of their acyclic numbers. When an intermediate vertex \( y \) survives the bounding test and is put in \( AN_i \), all modules in \( \mathbf{TG}(y) \) would have been scheduled and their completion times (and, thus, modified execution times) would have been determined in the bounding process in the previous stage (Step 3.3 in \( \mathbf{MA} \) in Section 3). Thus, when \( x \) is expanded from its parent vertex \( y \) by adding the new assignment of \( M_n \), the schedules, completion times and modified
execution times of all modules in \( \mathbf{TG}_i(y) \) (which includes all preceding modules of \( M_j \)) must have been determined. So, all the \( e_j \)'s needed in (4.3) are known at the time of calculating \( r_i \).

**Example 1.** Fig. 2 shows an example of how \( r_5 \) and \( LC_5 \) are calculated. The allocation \( x \) in Fig. 2 assigns \( M_1, M_5, M_6, M_8 \), and \( M_9 \) to \( N_1 \), and the other modules to \( N_2 \). Both module execution times and task deadlines are specified in the figure. The worst case IPC delay is assumed to be 0.5 unit of time, i.e., \( d_{ij} \) is 0.5 wherever applicable. For example, the release time \( r_5 \) of \( M_4 \) is calculated as

\[
r_5 = \max \{r_{r_5}, r_1 + \hat{e}_1 + \text{com}_{14}(x), r_2 + \hat{e}_2 + \text{com}_{24}(x), r_3 + \hat{e}_3 + \text{com}_{34}(x)\}
\]

\[= \max \{0, 0 + 1 + 0.5, 0 + 1, 1.5 + 2 + 0.5\} = 4,
\]

and the latest completion time, \( LC_{12} \), of \( M_{12} \) is calculated as

\[
LC_{12} = \min \{LC_{12}, LC_{13} - \epsilon_{13} - \text{com}_{1321}(x)\}
\]

\[= \min \{10, 12, -1\} = 10.
\]

Now, we describe the **MS** algorithm, the theoretical base of which is grounded on the result of [36]. First, we arrange the modules \( \in S_i(x) \) in the order of nondecreasing release times. We then decompose \( S_i(x) \) into blocks, where a block \( B \subset S_i(x) \) is defined as the minimal set of modules processed without any idle time from \( r(B) = \min_{M_{i} \in B} r_i \) until \( c(B) = r(B) + \epsilon(B) \), where \( \epsilon(B) = \sum_{M_j \in B} e_j \). That is, each \( M_i \notin B \) is either completed no later than \( r(B) \) or not released before \( c(B) \). For example, as shown in Fig. 3, the set of modules assigned to \( N_1 \) in Fig. 2, \( S_1(x) = \{M_1, M_5, M_6, M_8\} \), can be decomposed into three blocks, while the set of modules assigned to \( N_2 \), \( S_2(x) = \{M_2, M_4, M_9, M_{10}, M_{11}, M_{12}, M_{13}\} \), can be decomposed into two blocks.

Obviously, scheduling modules in a block \( B \) is irrelevant to that in other blocks, so we can consider each block separately. Let \( d_{ij} \) denote the outdegree of \( M_i \) within \( B \), i.e., the number of modules \( M_j \in B \) such that \( M_i \rightarrow M_j \). For each block \( B \), we first determine the set \( \hat{B} = \{M_i : M_i \in B, d_{ij} = 0\} \), i.e., modules without successors in \( B \), and then select a module \( M_m \) such that

\[
f_m(c(B)) = \min_{M_{i} \in B} f_i(c(B)), \tag{4.4}
\]

i.e., \( M_m \) has no successor within \( B \) and incurs a minimum cost if it is completed last in \( \hat{B} \). (In case of a tie, we choose the module with the largest acyclic number.) Now, consider an optimal schedule for the modules in \( B \) subject to the restriction that \( M_m \) is processed only if no other module is waiting to be processed. This optimal schedule consists of two parts:

**Sched1:** An optimal schedule with the cost \( f_{max}(B - \{M_m\}) \)
for the set \( B - \{M_m\} \), which could be decomposed into a number of subblocks \( \hat{B}_1, \hat{B}_2, \ldots, \hat{B}_n \).

**Sched2:** A schedule for \( M_m \), which is given by

\[
\left\lceil r(B), c(B) \right\rceil \bigcup_{j=1}^{n} \left\lceil r(\hat{B}_j), c(\hat{B}_j) \right\rceil,
\]

where \( r(B) = \min_{M_{i} \in B} r_i \) and \( c(B) = r(B) + \epsilon(B) \) with \( \epsilon(B) = \sum_{M_{i} \in B} e_j \).

For this optimal schedule, we have

\[
f_{max}(B) \text{ with the above restriction } = \max \left\{ f_m(c(B)), f_{max}(B - \{M_m\}) \right\} \leq f_{max}(B), \tag{4.5}
\]

where the last inequality comes from:

1) \( f_{max}(B) = \min_{M_{i} \in B} f_i(C_i) \geq \min_{M_{i} \in B} f_i(c(B)) = \min_{M_{i} \in B} f_i(c(B)) = f_m(c(B)) \) by the way \( \hat{B} \) was constructed from \( B \) and (4.4).

2) Since \( B - \{M_m\} \) is a subset of \( B \), \( f_{max}(B) \geq f_{max}(B - \{M_m\}) \), \( \forall M_m \).
It follows from (4.5) that there exists an optimal schedule in which $M_4$ is scheduled only if no other module is waiting to be processed. By repeatedly and recursively applying the above procedure to each of the subblocks $B_1, B_2, ..., B_p$, we obtain an optimal schedule for $B$. The rationale behind MS is that a PN is never left idle when there are modules ready to be processed, and, by virtue of the cost function defined, it is always the module $M_i$ with the smallest $LC_i$ that will be executed among all released modules.

**Example 2.** Take the task graph in Fig. 2 as an example, and consider the schedule on $N_1$. As shown in Fig. 3, $S_1(x)$ is composed of three blocks, $B_1 = \{M_1\}$, $B_2 = \{M_3\}$, and $B_3 = \{M_5, M_6\}$. The optimal schedule can be readily obtained (Fig. 3), since the schedules for $B_1$ and $B_2$ are trivial, and, for $B_3$, we have $\hat{B}_3 = \{M_6\}$, meaning that $M_6$ has to be processed only when no other module is waiting to be processed.

**Example 3.** Fig. 4 gives another (more complicated) illustrative example, showing how MS schedules the modules assigned to a PN $r_i$ and $LC_i$, $1 \leq i \leq 5$, are assumed to have been computed from the entire task graph and are given in the figure. By ordering the modules according to their increasing release times, we obtain two blocks: $B_1 = \{M_1, M_2, M_3, M_4\}$ from $[0, 8]$ (i.e., $r(M_1) = 0$, $r(M_2) = 8$, and $r(M_3) = 8$) and $B_2 = \{M_5\}$ from $[9, 11]$ (i.e., $r(M_5) = 9$, $r(M_6) = 2$, and $c(M_6) = 11$). The schedule for $B_2$ is trivial, because $B_2$ consists of a single module and itself represents an optimal schedule for $B_2$. For $B_1$ we have $\hat{B}_1 = \{M_3, M_4\}$ and select $M_3$ to be processed only when no other modules are waiting, since $LC_3 > LC_4$. Now, $B = \{M_5\}$ consists of two subblocks: $B_{11} = \{M_1, M_2\}$ from $[0, 3]$ and $B_{12} = \{M_4\}$ from $[4, 6]$. $B_{12}$ itself represents an optimal schedule. For $B_{11}$, we have $\hat{B}_{11} = \{M_1, M_2\}$ and select $M_1$ to be processed last, since $LC_1 > LC_2$. The final optimal schedule for $B_1$ is obtained by combining the optimal schedule for $B_{11}$ and $B_{12}$ ($Sched1$) and the schedule for $M_5$ ($Sched2$) which consists of $[0, 8] - [0, 3] \cup [4, 6]$. The result is depicted in the last row of Fig. 4.

**MS Algorithm:**

**Step 1:** Compute the latest completion time $LC_i$, $1 \leq i \leq N$, for $TG$. This computation requires $O(N^2)$ time.

**Step 2:** Compute the release time $r_i$ for $M_i \in TG(x)$ with respect to their precedence constraints. This computation, in the worst case, requires $O(N^3)$ time.

**Step 3:** Construct the blocks $B_1, B_2, ..., B_p$ of $S_1(x)$ for every $N_k$ by ordering the modules $\in S_1(x)$ according to their nondecreasing release times. This ordering requires $O(|S_1(x)| \cdot \log |S_1(x)|)$ time, $\forall k$.

**Step 4:** For each block $B_i$, $1 \leq i \leq b$, update the outdegree, $dg_j$, of every $M_j \in B_i$. This update requires $O(|S_1(x)|^2)$ time for all $B_i \subseteq S_1(x)$.
Step 5: For each block $B_i$, select $M_m \in B_i$ subject to (4.4), determine the subblocks of $B_i - \{M_m\}$, and construct the schedule for $M_m$ as given in Sched 2. Then, update the $d_{ij}$ of every $M_j \in B_i - \{M_m\}$ with respect to the subblock of $B_i - \{M_m\}$ to which $M_j$ belongs. By repeatedly applying Step 5 to each of the subblocks of $B_i - \{M_m\}$, one can obtain an optimal schedule. The time complexity for all repeated applications of Step 5 is bounded by $O(\lvert S_2(x) \rvert^3)$.

Since the time complexity associated with each step is polynomial, the MS algorithm is a polynomial algorithm.

4.2 Calculation of $P_{ND}(x)$

We are now in a position to discuss how to calculate $P(T_{\ell})$ timely completed under $x$. Conceptually, given $TG$ and $x$, we can determine the set, $S_2(x)$, of modules $\in TG$ assigned to $N_\ell$ and, then, use MS to schedule modules in $S_2(x)$, $\forall k$. The completion time(s) of the last module(s) in $T_{\ell} \cap TG$ under these schedules determines whether $T_{\ell}$ can be completed in time or not. However, since $TG$ may contain loops and/or OR-subgraphs, the release times and the latest completion times of modules needed in Step 3 of MS may not be readily determined. Moreover, one cannot determine which module of $T_{\ell}$ to execute last if the last component in $T_{\ell}$ is an OR-subgraph.

4.2.1 Component Graphs

To resolve the above problem, we must eliminate the loops/OR-subgraphs in $TG$ while retaining all the timing and probabilistic properties of $TG$. We first calculate the latest completion time, $LC_t$, of $M_i \in TG$ using (4.2), assuming that

A1. Every OR-subgraph following $M_{ii}$ if any, is viewed as an AND-subgraph by ignoring branching probabilities.

A2. Every loop $L_n$ following $M_{ii}$ if any, is replaced by a cascade of $n_{ii}$ copies of its loop body, where $n_{ii}$ is the maximum loop count.

With A1 and A2, the $LC_t$ calculated are the worst-case latest completion time.

Second, we represent each loop $L_n \in TG$ with the cascaded $m$ copies of its loop body with probability $(1 - q_n)q_n^{m-1}$, where $1 \leq m \leq n_{ii}$, and $q_n$ is the looping-back probability of $L_n$. The last copy of $M_i \in L_n$ bears the $LC_t$ calculated above, while the $(n_{ii} - j)$th copy of $M_i$ bears the latest completion time $LC_t - j \cdot e(L_n)$, where $e(L_n)$ is the execution time of the loop body. Also, we represent each OR-subgraph $O_{ij}$ in $TG$ with its $n_{ij}$th branch with probability $q_{n_{ij}n_{ij}}$, where $1 \leq n \leq n_{ij}$, and $q_{n_{ij}n_{ij}}$ is the branching probability of the $n_{ij}$th branch of $O_{ij}$ and $n_{ij}$ is the number of branches in $O_{ij}$.

The $TG$ can then be represented by the set of all possible combinations—which is termed as the set of component task graphs. For example, if there exists a loop $L_n$ and an OR-subgraph $O_{ij}$ in $TG$, then there are a total of $n_{ii} \times n_{ij}$ component graphs of $TG$, and with probability $p_c = (1 - q_n)q_n^{m-1} \cdot q_{n_{ij}n_{ij}}$, the $TG$ is represented by the $TG_c$ with $L_n$ replaced by the cascaded $m$ copies of its loop body and $O_{ij}$ replaced by its $n_{ij}$th branch. (One can trivially extend this to the case where there is more than one loop and/or OR-subgraph.)

For each component graph $TG_{ij}$, of $TG$, we then calculate the release time, $r_{ij}$, of $M_i \in TG$, using (4.3). Using the $r_{ij}$ and $LC_t$s determined above, we can apply Steps 3-5 in $MS$ to find the best schedules for all modules in $TG_{ij}$. Note that, in a component graph $TG_{ij}$, the release time, $r_{ij}$, and the number of times $M_i$ is executed are both fixed, making it possible to decompose $S_2(x)$ into blocks.

For real-time applications in which the worst-case performance is the main concern or, for a $TG$ which contains a large number of loops and/or OR-subgraphs, we can represent the $TG$ with the set of single component task graphs in which each loop $L_n$ is replaced by the cascaded $n_{ii}$ copies of the loop body (while each OR-subgraph is replaced by one of its branches). This significantly reduces the number of component graphs needed to be considered.

4.2.2 Calculation of $P_{ND}(x)$

We now calculate, for every $T_{\ell}$ in a component task graph $TG_{ij}$, the probability $P(T_{\ell} \text{ is timely completed under } x)$. Let the critical time of $M_i \in TG_{ij} \neq 0$ defined as the latest time $M_i$ should be completed for the timely completion of the task $T_{\ell}$ only. Note that $D_i$ can be obtained in the same way as $LC_{ij}$ except that the precedence relations, $M_i \rightarrow M_j$, when $M_i \notin TG_{ij}$, are ignored. That is, let $D_i$ be initially set to the deadline of $T_{\ell}$ to which $M_i$ belongs. Then, $D_i$ is modified as:

$$D_i = \min \left\{ D_j, \min \left\{ D_{ij} - e_{ij} - \text{com}_{ij}(x) : M_i \rightarrow M_j \right\} \right\},$$

where $i = N - 1, N - 2, \ldots, 1$.

Obviously, $D_i \geq LC_{ij}$. Also, let $\hat{T}_{\ell} = \{ M_i : M_i \in T_{\ell} \cap TG_{ij}, d_{ij} = 0 \text{ with respect to } T_{\ell} \cap TG_{ij} \}$ be the set of modules without any successor in $T_{\ell} \cap TG_{ij}$. Then, $T_{\ell}$ can be timely completed under the allocation of $x$ in the component task graph $TG_{ij}$ if $D_i \geq C_i$ for every module $M_i$ which has no successor in $T_{\ell} \cap TG_{ij}$ (i.e., $\forall M_i \notin \hat{T}_{\ell}$). In other words, the probability $P(T_{\ell} \text{ is timely completed under } x \text{ in } TG_{ij})$ can be expressed as:

$$P(T_{\ell} \text{ is timely completed under } x \text{ in } TG_{ij}) = \prod_{M_i \in \hat{T}_{\ell}} \delta(D_i - C_i),$$

where $\delta()$ is the step function, i.e., $\delta(t) = 1$ for $t \geq 0$, and $\delta(t) = 0$, otherwise. Consequently, $P(T_{\ell} \text{ is timely completed under } x)$ can be expressed as:

$$P(T_{\ell} \text{ is timely completed under } x) = \sum_{\text{all } TG_{ij}} p_c \cdot P(T_{\ell} \text{ is timely completed under } x \text{ in } TG_{ij}),$$

where $p_c$ is the probability that $TG$ is represented by $TG_{ij}$. Finally, $P_{ND1}$ can be expressed as:

$$P_{ND1} = \sum_{\text{all } TG_{ij}} p_c \cdot P(T_{\ell} \text{ is timely completed under } x \text{ in } TG_{ij}).$$
\[ P_{\text{ND2}}(x) = \prod_{\ell=1}^{N_{T}} P(T_{\ell} \text{ is timely completed under } x), \] (4.8)

where \( N_{T} \) as defined in Section 2, is the number of periodic tasks in the system.

5 Evaluation of Logical Correctness

In this section, we calculate the probability, \( P_{\text{ND2}}(x) \), that:

1) All PNs are operational during the execution of modules assigned to them, and
2) All communication links are operational during the course of IPCs.

The derivation of \( P_{\text{ND2}}(x) \) is similar to that of the reliability function in [18], [19], but we relax the following two unrealistic assumptions used in [18], [19]:

A1) The network topology is cycle-free, i.e., there is one and only one path between any pair of PNs;
A2) Each module is executed only once in a task invocation.

Instead of the first assumption, we allow an arbitrary network topology, and also allow the IPCs between \( N_{k} \) and \( N_{\ell} \) to take place over one of the (arbitrarily chosen) edge-disjoint paths between the two PNs. In contrast to the second assumption, we allow modules to be contained in loops and/or branches of OR-subgraphs, i.e., modules may be executed more than once, or not executed at all in a task invocation.

To facilitate the derivation of \( P_{\text{ND2}}(x) \), we need the following notation:

- LP: the set of modules which are contained in loops.
- OR: the set of modules which are on the branches of OR-subgraphs.
- \( q_{\ell} \): the looping-back probability of loop \( L_{\ell} \).
- \( q_{O\ell} \): the branching probability of the \( \ell \)th branch of an OR-subgraph, \( O_{\ell} \).
- \( n_{L_{\ell}} \): the maximum count of loop \( L_{\ell} \).
- \( n_{O_{\ell}} \): the number of branches in an OR-subgraph \( O_{\ell} \).
- \( \lambda_{k} \): the constant exponential failure rate of \( N_{k} \).
- \( \hat{\lambda}_{mn} \): the constant exponential failure rate of link \( \ell_{mn} \). Failure occurrences are assumed to be statistically independent of one another.

\[ R_{im}(i, j, n_{e}, x) \text{: the probability that link } \ell_{mn} \text{ is operational during the } n_{e} \text{ occurrences of IMC between } M_{i} \text{ and } M_{j} \text{ under allocation } x. \]

\[ R_{im}(i, j, x) : \text{the probability that all PNs are operational during the execution of modules assigned to them under } x. \]

\[ R_{iub}(x) : \text{the probability that all links are operational for all IMCs under } x. \]

Under allocation \( x \), the probability that all PNs remain fault-free during the execution of the modules assigned to them is:

\[ R_{p_{im}}(x) = \prod_{M_{i} \in L_{\text{LP}}(x)} \prod_{k=1}^{K} \exp(-\lambda_{k} x_{\ell} \epsilon_{k}) \cdot \prod_{L_{\ell} \in L_{\text{LP}}(x)} \left( \prod_{\ell=1}^{n_{L_{\ell}}} (1 - q_{\ell}) q_{\ell}^{-1} \prod_{M_{i} \in L_{\ell}} \prod_{k=1}^{K} \exp(-\lambda_{k} x_{\ell} \epsilon_{k}) \right) \cdot \prod_{O_{\ell} \in \text{OR}} \left( \prod_{\ell_{mn} \in \text{branch of } O_{\ell}} \prod_{k=1}^{K} \exp(-\lambda_{k} x_{\ell} \epsilon_{k}) \right). \] (5.1)

Note that all factors, except the one associated with \( x_{\ell} = 1 \) in the term \( \prod_{K} \exp(-\lambda_{k} x_{\ell} \epsilon_{k}) \), reduce to one. The expression within the first pair of braces is the probability that the PNs on which stand-alone modules of each module are operational during the execution of these modules. Similarly, the expression in the second (third) pair of braces is the probability that the PNs on which the modules in loops (OR-subgraphs) reside are operational during the execution of these modules. In case \( M_{i} \) is contained in a loop \( L_{\ell} \) with probability \( q_{\ell}^{-1}(1 - q_{\ell}) \), \( M_{i} \) requires an execution time \( \ell \cdot \epsilon_{\ell} \) and, in case \( M_{i} \) is on the \( \ell \)th branch of an OR-subgraph, \( O_{\ell} \), with probability \( q_{O\ell} \), \( M_{i} \) will be executed. Note that (5.1) can be readily extended to the case where a loop/OR-subgraph is contained in other loops and/or OR-subgraphs.

Example 4. Consider Fig. 1b as an example, where \( L_{\text{LP}} = \{ M_{3} \} \) and \( \text{OR} = \{ M_{6}, M_{5} \} \). If all modules of \( T_{1} \) are assigned to \( N_{1} \) and all modules of \( T_{2} \) and \( T_{3} \) are assigned to \( N_{2} \), i.e., \( x_{11} = x_{31} = x_{41} = x_{71} = 1 \) and \( x_{22} = x_{32} = x_{42} = x_{52} = x_{62} = x_{102} = x_{112} = 1 \), then

\[ R_{pm}(y) = e^{-\lambda_{t} \epsilon_{t} (t + e_{t})} \cdot e^{-\lambda_{f} \epsilon_{f} (t + e_{f} + e_{t} + e_{p} + e_{i})} \cdot \left( q_{e} e^{-\lambda_{f} \epsilon_{f}} + (1 - q_{e}) e^{-\lambda_{f} \epsilon_{f}} \right) \cdot \prod_{i=1}^{n_{f}} (1 - p) p^{\ell-1} e^{-\lambda_{f} \epsilon_{f}}. \]

The expression of \( R_{pim}(x) \) calls for the derivation of the probability that link \( \ell_{mn} \) is operational during the \( n_{e} \) occurrences of IMC between \( M_{i} \) and \( M_{j} \) under \( x \), \( R_{mn}(i, j, n_{e}, x) \): \( R_{mn}(i, j, n_{e}, x) = \prod_{k=1}^{K} \prod_{\ell=1}^{n_{L_{\ell}}} \prod_{\ell_{mn} \in \text{edge-disjoint paths}} \exp\left(-\hat{\lambda}_{mn} \cdot \frac{n_{e} \lambda_{mn} d_{ij}}{n(k, \ell)} \cdot x_{\ell} \epsilon_{\ell} \cdot l(m, n, k, \ell)\right). \] (5.2)

Two remarks are in order:

- All the \( K(K - 1) \) terms in (5.2)—except for the term corresponding to \( x_{\ell} = x_{\ell'} = 1 \)—reduce to one.
- If \( M_{i} \) is assigned to \( N_{x} (x_{3} = 1) \), \( M_{i} \) is assigned to \( N_{\ell} \) \( (x_{\ell} = 1) \), and \( \ell_{mn} \) lies on one of the edge-disjoint paths between \( N_{x} \) and \( N_{\ell} \) \( (l(m, n, k, \ell) = 1) \), then

\[ R_{m,n}(i, j, n_{e}, x) = \exp\left(-\hat{\lambda}_{mn} \cdot \frac{n_{e} \lambda_{mn} d_{ij}}{n(k, \ell)}\right), \]

where \( \frac{n_{e} \lambda_{mn} d_{ij}}{n(k, \ell)} \) is the (average) worst case communication

6. Modules which are contained in neither loops nor OR-subgraphs in the TG.
time over link $\ell_{mn}$ contributed by the $n_c$ occurrences of IMC between $M_i$ and $M_j$.

**Example 5.** Given the simple distributed system represented by a complete graph of three PNs (where $n(k, \ell) = 2$ and $l(m, n, k, \ell) = 1$), $1 \leq m, n \leq 3$, $m \neq n$, $1 \leq k, \ell \leq 3$, $k \neq \ell$) and the TG in Fig. 1b, we have

$$R_{12}(i, j, n, x) = \exp \left( -\hat{\lambda}_{12} \cdot \frac{n l_{12} d_{ij}}{2} \right);$$

since $x_1 x_2 = 1$, and $R_{12}(2, 4, n, x) = 1$, i.e., the IMCs between $M_2$ and $M_4$ are accomplished via shared memory and do not use link $\ell_{12}$ at all.

Now, we are in a position to derive the expression of $R_{\text{link}}(x).$ Let

$$C_1 = \{ (M_i, M_j) : d_{ij} > 0, \text{ and neither of } M_i \text{ and } M_j \text{ reside in a loop or an OR - subgraph} \};$$

$$C_2 = \{ (M_i, M_j) : d_{ij} > 0, \text{ and one of } M_i \text{ or } M_j \text{ resides in a loop while the other (not contained in an OR - subgraph) resides immediately before or after the loop} \};$$

$$C_3(L_a) = \{ (M_i, M_j) : d_{ij} > 0, \text{ and both } M_i \text{ and } M_j \text{ reside in the loop } L_a \}, \forall L_a \in LP;$$

$$C_4(O_{b}, \ell) = \{ (M_i, M_j) : d_{ij} > 0, \text{ and either } M_i \text{ or } M_j \text{ or both reside on the } \ell \text{ th branch of the OR-subgraph } O_{b} \}, \forall O_{b} \in OR, 1 \leq \ell \leq n_{O_{b}}.$$ 

where $d_{ij} > 0$ indicates that $M_i$ and $M_j$ communicate with each other. The rationale behind defining $C_1, C_2, C_3(L_a),$ and $C_4(O_{b}, \ell)$ is as follows: If $(M_i, M_j) \in C_1 \cup C_2$, then $M_i$ and $M_j$ communicate with each other exactly once. If $(M_i, M_j) \in C_3(L_a)$, then $M_i$ and $M_j$ communicate $n_c$ times 

$$R_{\text{link}}(x) = \prod_{\ell_{mn}} \prod_{(M_i, M_j) \in C_{\ell_{mn}}} R_{mn}(i, j, 1, x);$$

$$\prod_{l \in L_{R}} \left( \sum_{m=1}^{n_{c}} q_{s}^{-1}(1 - q_{s}) \cdot \prod_{(M_i, M_j) \in C_{l_{mn}}} R_{mn}(i, j, 1, x) \right).$$

For clarity of presentation, (5.3) excludes the case where a loop/OR-subgraph is contained in other loops and/or OR-subgraphs. However, it is straightforward to extend (5.3) to include such a case.

**Example 6.** Consider again the example of allocating the TG in Fig. 1b to the distributed system represented by a three-complete graph. We have $C_1 = \{ (M_1, M_4), (M_1, M_5), (M_2, M_4), (M_2, M_5), (M_3, M_4), (M_3, M_5), (M_4, M_5), (M_5, M_1), (M_{10}, M_{11}) \}$, $C_2 = \{ (M_4, M_5), (M_5, M_9) \}$, $C_3 = \emptyset$, $C_4(O_{1}, 1) = \{ (M_1, M_4) \}$, and $C_4(O_{2}, 2) = \{ (M_1, M_2) \}$. Thus,

$$R_{\text{link}}(x) = \prod_{\ell_{mn}} \prod_{(M_i, M_j) \in C_{\ell_{mn}}} R_{mn}(i, j, 1, x);$$

$$\prod_{l \in L_{R}} \left( \prod_{m=1}^{n_{c}} q_{s}^{-1}(1 - q_{s}) \cdot \prod_{(M_i, M_j) \in C_{l_{mn}}} R_{mn}(i, j, 1, x) \right).$$

Under the allocation $x$ given in Example 5, we have

$$R_{\text{link}} = \prod_{\ell_{mn}} R_{mn}(1, 4, 1, y) \cdot \prod_{\ell_{mn}} R_{mn}(4, 5, 1, y) \cdot e^{-\hat{\lambda}_{12} t_{12} + \hat{\lambda}_{13} t_{13} + \hat{\lambda}_{23} t_{23}} \int_{t_{12}} \cdot e^{-\hat{\lambda}_{12} t_{12} + \hat{\lambda}_{13} t_{13} + \hat{\lambda}_{23} t_{23}} \int_{t_{13}} \cdot e^{-\hat{\lambda}_{12} t_{12} + \hat{\lambda}_{13} t_{13} + \hat{\lambda}_{23} t_{23}} \int_{t_{32}},$$

where the first and the second factors are contributed by the IMC between $M_1$ and $M_4$ and between $M_4$ and $M_5$, respectively. For example, $e^{-\hat{\lambda}_{12} t_{12} d_{12}}$ is contributed by the IPC between $M_1$ and $M_4$ which runs through $\ell_{12}$ and $e^{-\hat{\lambda}_{12} t_{12} d_{12}} \hat{\lambda}_{13} t_{13}$ is contributed by the IPC which routes through $\ell_{13}$ and $\ell_{32}$.

Finally, we have

$$P_{NPD}(x) = R_{pm}(x) \cdot R_{\text{link}}(x).$$

**6. Branching and Bounding Tests**

The branching test uses the dominance relation derived from the requirement of timely completion of tasks to limit the number of child vertices generated in the branching process. The bounding test derives an UBOF for each intermediate vertex with which one decides whether or not to prune an intermediate vertex in the bounding process. In this section, we discuss how we design the branching and bounding tests.

**6.1 Branching Test**

Recall that expanding an intermediate vertex $x$ in the search tree corresponds to allocating the module with the smallest acyclic number that has not yet been allocated (i.e., a module in $TG_{\setminus T(x)}$). The branching test uses the following...
dominance relation. \( M_i \) can be invoked after all its precedence constraints are met and must be completed by its latest completion time, \( LC_p \), to ensure that all its succeeding tasks meet their deadlines. Hence, if

1. the idle time of a PN, say \( N_{ik} \), during the interval \([r_p, LC_p]\) is smaller than \( e_i \), and
2. the module, say \( M_i \), scheduled to be executed\(^7\) on \( N_{ik} \) in \([r_p, LC_p]\) under a partial allocation \( x \) has tighter timing constraints than \( M_i \) (so no preemption on \( N_{ik} \) is possible to ensure the completion of \( M_i \) before \( LC_p \)),

then allocating \( M_i \) to \( N_{ik} \) is likely to miss \( M_i \)'s latest completion time. Thus, \( N_{ik} \) should not be a candidate PN for allocating \( M_i \), i.e., \( N_{ik} \) fails the branching test.

**Branching Test:**

**Step 1.** Calculate optimistic estimates, \( r_i^o \) and \( LC_i^o \), of \( r_i \) and \( LC_i \), assuming that

A1. Every pair of communicating modules that have not yet been assigned (i.e., \( \in TG\setminus TG(x) \)) on the same PN.

A2. The OR-subgraph preceding or following \( M_i \), if any, is replaced by the branch with the smallest flowtime.

A3. The loop preceding, containing, or following \( M_i \) if any, is replaced by its loop body (i.e., the loop executes only once).

**Step 2.** Calculate an pessimistic estimate, \( LC_i^\gamma \), of \( LC_i \), assuming that

A4. The IMCs in \( TG\setminus TG(x) \) are executed on \( N_k \) with the largest nominal inter-PN delay \( y_{tk} \).

A5. The OR-subgraph following \( M_i \) is replaced by the branch with the largest flowtime.

A6. The loop following \( M_i \), if any, is replaced by the cascaded \( n_t \) copies of its loop body, where \( n_t \) is its maximum loop count.

**Step 3.** For each \( N_{ik} \), check whether the following two conditions are true or not:

C1. The idle time of \( N_{ik} \) in \([r_i^o, LC_i^o]\) is less than \( e_i \).

C2. \( LC_j \leq LC_i^\gamma \), where \( LC_j \) is the latest completion time of the module, \( M_j \), scheduled last in \([r_i^o, LC_i^o]\) on \( N_{ik} \) under the partial allocation \( x \).

If both conditions are true, then \( N_{ik} \) fails the test and is not considered for allocating \( M_i \).

A1-A3 ensure \( r_i^o \leq r_i \) and \( LC_i^o \geq LC_i \), thus making the interval \([r_i^o, LC_i^o]\) larger than \([r_i, LC_i]\). A4-A6 ensure \( LC_i^\gamma \leq LC_i \), making \( M_i \) likely to preempt other modules on \( N_{ik} \). Consequently, the use of the optimistic interval, \([r_i^o, LC_i^o]\), and the pessimistic value, \( LC_i^\gamma \), ensure that the PNs which fail the branching test cannot indeed complete \( M_i \) in time.

6.2 Calculation of an UBOF for the Bounding Test

The bounding test calculates an UBOF for each intermediate vertex, and prunes (keeps) the intermediate vertex when the calculated UBOF \( \leq (>) \) the best objective function, \( P^{*'}_{ND} \), found thus far. The bounding test uses the following principles. A vertex \( y \) is generated from its parent vertex \( x \) by adding the assignment \( M_i \to N_k \) to the partial allocation \( x \) for some \( N_k \) that survives the branching test. After including \( M_i \to N_k \), \( N_k \) needs to reschedule the modules assigned to it under \( y \) (i.e., the modules \( \in S_i(y) \) using MS. Because modules are assigned in acyclic order, all preceding modules of \( M_i \) in \( S_i(y) \) have their latest completion times \( < LC_i \), and their schedules will not be changed by the addition of \( M_i \). On the other hand, if some nonpreceding module \( M_j \) makes its schedule as a result of \( M_i \to N_k \), then the release time(s) of all \( M_j \)'s succeeding module(s) have to be changed accordingly. Consequently, the PNs (\( \neq N_k \)) on which these succeeding modules of \( M_i \) reside need to reconsider their module schedules.

**Example 7.** Fig. 5 gives an example of how adding \( M_i \to N_k \) to a partial allocation might affect the schedules on other PNs. In the partial allocation \( x \) prior to the assignment \( M_{ik} \to N_k \), \( M_1 \) and \( M_2 \) are assigned to \( N_1 \) and \( N_2 \) in (4.6) is modified as

\[
\begin{align*}
\hat{T}_i &= \{M_j : M_j \in T_i \cap TG_i(y), d_{ij} = 0 \text{ w.r.t. } T_i \cap TG_i(y)\}, \\
\hat{T}_k &= \{N_{ik} : N_{ik} \in TG(y) \setminus TG(x)\}.
\end{align*}
\]

Let \( PN \) denote the set of PNs which need to reconsider their module schedules as a result of \( M_i \to N_k \). Then, a UBOF is calculated by the following steps.

**Calculation of a UBOF:**

**Step 1.** Represent the TG with the set of component graphs.

**Step 2.** Calculate \( \hat{P}_{NDI}(y) \) as follows:

**Step 2.1.** In each component graph \( TG_i \):

**Step 2.1.1.** Reschedule the modules \( \in S_i(y) \) for every \( N_{im} \in PN \) using MS and A1 in the branching test.

**Step 2.1.2.** Use (4.6) to calculate \( P(T_i \text{ is timely completed under } y \in TG_i) \) for every \( T_i \in TG_i \), where \( \hat{T}_i \) in (4.6) is modified as A7.

\[
\hat{T}_i \setminus \{M_j : M_j \in T_i \cap TG_i(y), d_{ij} = 0 \text{ w.r.t. } T_i \cap TG_i(y)\}.
\]

**Step 2.2.** Calculate \( P(T_i \text{ is timely completed under } y) \) for
every \( T' \in \text{TG} \) and \( \hat{p}_{\text{ND}1}(y) \), using (4.7) and (4.8), respectively.

**Step 3.** Calculate \( \hat{p}_{\text{ND}2}(y) \) using (5.1), (5.3), and (5.4), and A8. Every \( M_i \in \text{TG}\setminus\text{TG}(y) \) is assumed to be allocated to the most reliable PN, and every pair of communicating modules in \( \text{TG}\setminus\text{TG}(y) \) reside on the same PN.

**Step 4.** Calculate \( \hat{p}_{\text{ND}}(y) = \hat{p}_{\text{ND}1}(y) \cdot \hat{p}_{\text{ND}2}(y) \).

Note that, because of the use of A1 and A7, \( \hat{p}_{\text{ND}1}(y) \) derived above is an upper bound of \( p_{\text{ND}1} \) of any leaf vertex (complete allocation) generated from \( y \). Moreover, whether or not modules in \( \text{TG}\setminus\text{TG}(y) \) meet their latest completion times need not be considered in the calculation of \( \hat{p}_{\text{ND}1}(y) \), and is thus excluded by A7.

### 7 Numerical Examples

The performance of MA is evaluated according to the following sequence:

1) discussion on the generation of task graphs and distributed systems;
2) the characteristics of MA;
3) the practicality of MA.

#### 7.1 Generation of Task Graphs and Distributed Systems

There are a large number of parameters that may affect the performance of MA. They can be classified as system parameters, which specify the distributed system under consideration, and task parameters, which specify the TG. The generation of realistic TGs and distributed systems largely depends on how these parameters are specified. However, little is reported in the literature about “typical” real-time TGs and their communication patterns. Thus, we randomly generate both system and task parameters in our numerical experiments. We believe these randomly generated task graphs cover a wide spectrum of real-time applications.

The number of PNs in the distributed system is varied from three to 40. After the number of PNs is selected, the network topology is then arbitrarily generated. The worst case link delay, \( t_{\text{max}} \), associated with \( t_{\text{max}} \) is exponentially distributed with mean 0.1, \( e \), where \( e \) is the mean module execution time. The node failure rate, \( \lambda_n \), and the link failure rate, \( \lambda_{\text{mn}} \), were varied from \( 10^{-5} \) to 0.5 (1/time unit). The number of modules, \( N \), to be allocated is varied from four to 50. The execution time of a module is exponentially distributed with mean 1.0 unit of time. The IPC volume between two communicating modules is uniformly distributed over \( (0, 10] \) data units. The precedence constraints and the timing requirements of the TG are also randomly generated.

Before running experiments, we eliminated the TGs which were definitely infeasible. Infeasibility is detected by...
calculating release times and latest completion times of all modules, while ignoring all IMC times. If the interval between the latest completion time and the release time is less than the execution time for some module(s) in all the component graphs of a TG, this TG is infeasible (in the sense that some tasks cannot be completed in time even if infinite resources were available) and is not considered any further.

All experiments were performed on a Sun4 SPARC station running the SUNOS 4.1.3 operating system. Due to space limitation, we present only a few representative cases and statistical results. However, the conclusions drawn from the following summary were corroborated by all the experiments conducted.

7.2 Characteristics of MA

By virtue of the BB method, MA always yields the best allocation. To further examine the characteristics of the optimal allocation found by MA, experiments were performed on

1) TGs with different degrees of parallelism;
2) task sets with different degrees of deadline tightness;
3) distributed systems with different worst case link delays and node/link failure rates.

Several interesting properties observed in the experiments are given below.

P1. MA allocates sequentially-executing modules subject to the same tight timing constraints to the same PN. For example, the best allocation of the TG in Fig. 1b to the distributed system represented by a complete graph of three PNs and with homogeneous node failure rates ($\lambda_k = 0.001$) and link failure rates ($\lambda_{mn} = 0.001$) is to assign $T_1$ to $N_1$, and both $T_2$ and $T_3$ to $N_2$. MA recognizes that the execution path $M_2 \rightarrow M_1 \rightarrow M_4 \rightarrow M_8 \rightarrow M_9 \rightarrow M_{10}$ to $M_3$ in the TG is critical subject to $T_i$'s deadline and cannot tolerate any IPC delay, thus allocating both $T_2$ and $T_3$ to the same PN. The resulting best objective function value is $P_{ND} = 9.8227 \times 10^{-1}$.

P2. Heavily communicating modules may not necessarily be allocated to the same PN. For example, consider the allocation of the TG in Fig. 6a. The attributes of both the TG and the distributed system are specified in Experiment I. As shown in Fig. 6b, MA allocates $M_1$, $M_4$, and $M_9$ to $N_1$; $M_5$, $M_6$, and $M_{10}$ to $N_2$, $M_2$, and $M_3$ to $N_3$ so that all modules meet their latest completion times ($P_{ND1} = 1.0$) and are allocated to the most reliable PNs, $N_1$ to $N_3$ ($P_{ND2} = 9.7933 \times 10^{-3}$). Although the IMC between $M_4$ and $M_9$ is twice more than the others, $M_4$ and $M_9$ are allocated to different PNs. This is mainly because $T_2$ has a less tight timing constraint than others and can thus allow IPCs among its modules. This observation is in sharp contrast to the common notion that heavily communicating modules should always be co-allocated [23], [37].

P3. If the distributed system is homogeneous, MA assigns modules, subject to task timing constraints, in such a way that as few IPCs as possible will occur. To demonstrate this tendency, consider Experiment II in Fig. 6. The attributes of both the TG and the distributed system remain the same as in Experiment I, except that $\lambda_4 = 0.001$ (i.e., the distributed system becomes homogeneous). Now, the allocation and schedules specified in Fig. 6c give the best solution ($P_{ND1} = 1.0, P_{ND2} = 9.8096 \times 10^{-3}$). Note that the only IPC occurs between $M_1$ and $M_4$, which cannot be eliminated, because all modules of $T_1$ cannot be allocated to the same PN under $T_i$'s timing constraint.

P4. If both timeliness and logical correctness cannot be achieved at the same time, MA maximizes $P_{ND}$ by making a compromise between these two objectives. This is demonstrated by conducting three experiments: in Experiment I, the deadlines of the four tasks are set as $d_1 = 5.0, d_2 = 6.0, d_3 = 6.0$, and $d_4 = 8.0$. As shown in Fig. 6b, MA allocates modules only to (three) reliable PNs while meeting the timing constraints ($P_{ND1} = 1.0, P_{ND2} = 9.7933 \times 10^{-3}$). As the deadline constraints get tighter, in Experiment III, $d_1$ and $d_2$ remain unchanged while $d_3$ becomes 4.0 and $d_4$ becomes 7.5, MA is “forced” to allocate some of the modules ($M_i$) to a less reliable PN ($N_i$) in order to meet all timing constraints. Fig. 6c gives the best allocation and schedules: $T_2$ and $T_3$ are now allocated to $N_1$, $T_4$ to $N_2$, and $T_1$ to $N_3$ and $N_4$ ($P_{ND1} = 1.0, P_{ND2} = 9.6346 \times 10^{-3}$). On the other hand, if $N_4$ is highly prone to failure, as is assumed in Experiment IV ($\lambda_4$ is increased from 0.01 to 0.5), MA decides not to use $N_4$ at the risk of not making task $T_i$'s deadline, as depicted in Fig. 6d ($P_{ND1} = 0.7, P_{ND2} = 9.8096 \times 10^{-3}$, and $P_{ND} = 6.8667 \times 10^{-4}$).

Note that, for ease of exposition, we choose the simple TG given above as the example task system. However, the observation made was corroborated by all the experiments conducted. See [38] for more experimental results.

7.3 Practicality of MA

To test the practicality of MA for reasonably large TGs and/or distributed systems, we ran experiments on

1) TGs with 4-50 modules, while varying module execution times, IMC volumes, task deadlines, and randomly generating precedence constraints;
2) distributed system topologies with 3-40 PNs, while randomly varying worst case link delays and the degree of network connectivity.

We then computed the ratio of the number of search-tree vertices visited to the total number, $K^N+1 - 1$, of vertices in the search tree. The numerical results for different combinations of $N$ and $K$ are summarized in Table 1. The number of trials in each combination of $N$ and $K$ was determined,

8. A TG is said to have a high-degree of parallelism if most of its modules can be executed in parallel when there are enough resources. This could occur if the TG contains AND-subgraphs with a large number of branches and/or most tasks in the TG do not communicate with one another so that only a few precedence constraints are imposed on the modules belonging to different tasks.

9. Under the assumption that the parameter to be estimated (i.e., the mean number of search-tree vertices visited) has a normal distribution with

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so that a 95 percent (90 percent) confidence level may be obtained for a maximum error within 10 percent of the average numbers reported for $N \leq 10$ ($N > 10$). Also given in each combination are the worst and best results ever found in these trials.

In all the experiments conducted, no more than 9 percent of the search-tree vertices were visited before finding the best allocation for $N \geq 6$ and $K \geq 3$. Also, the percentage of search-tree vertices visited falls drastically as $N$ and/or $K$ grows, as shown in Table 1. This is because the “increasing rate” for the number of vertices visited as $N$ and/or $K$ grows is far lower than exponential. This suggests that both the dominance relation and the UBOF derived effectively prune unnecessary search paths at early stages of the BB process.

According to our experimental experiences, however, it takes a significant amount of CPU run time (usually over six hours on a SPARC Sun4 SPARC station running the SUNOS 4.1.3 operating system) for a single experiment for $N \geq 40$ and $K \geq 30$, which makes collecting statistics difficult (although obtaining the best allocation for a single experiment is still possible). Some new techniques may be needed to reduce the search space. For example, based on the observation $P_1$, one can co-allocate sequentially-executing modules in the TG subject to the same timing constraints. This can be done by calculating the release times and the latest completion times of all modules, while ignoring all IPC delays. If some module, $M_i$, has $L'_{Ci} - r_i$ equal to $e_i^{10}$ in all the component graphs of the TG, then this module and both its preceding and succeeding modules which are subject to the same timing constraint should be co-allocated.

### 8 Conclusion

We have addressed the problem of allocating periodic task modules in a distributed real-time system subject to precedence constraints, timing requirements, and intermodule communications. The probability of no dynamic failure is used as the objective function to incorporate both the timeliness and logical correctness of real-time tasks/modules into module allocation. $MA$ not only assigns modules to PNs, but also uses $MS$ to schedule all modules assigned to each PN.

An interesting finding from the numerical experiments is that $MA$ tends to allocate sequentially-executing modules subject to the same timing constraints to the same PN. Also, the common notion in general-purpose distributed systems that heavily communicating modules should be co-located [37], [23] may not always be applicable to real-time systems. Only in case there are enough resources to meet the timing requirement in a homogeneous distributed system, $MA$ assigns modules so as to minimize IPCs. Based on a set of experiments using randomly-generated TGs and distributed systems, $MA$ has also been shown to be computationally tractable for $N \leq 50$ and $K \leq 40$.

Despite its advantages mentioned above, $MA$ still takes a significant amount of time to locate an optimal allocation for the case of $N \geq 40$ and $K \geq 30$, due to the fact that there exist an extremely large number of search paths which might lead to an optimal solution and, thus, cannot be pruned at early stages of the BB process. One challenging extension to this research is to investigate the problem of grouping modules and/or PNs to reduce the size of the search space without resorting to a heuristic-directed technique. The conditions under which modules could be co-allocated, e.g., $P_1$ and $P_2$ observed in Section 7, are currently explored further, and will be reported in a forthcoming paper.
Fig. 6. An example showing how MA allocates modules: (a) The task graph and the system configuration used, (b) allocation and PN schedules for Experiment I: $M_1$, $M_5$, and $M_7$ are assigned to $N_1$; $M_6$, $M_8$, and $M_{10}$ are assigned to $N_2$; $M_2$ and $M_3$ are assigned to $N_3$, (c) allocation and PN schedules for Experiment II and III: $M_4$, $M_5$, $M_6$, and $M_7$ are assigned to $N_1$; $M_8$, $M_9$, and $M_{10}$ are assigned to $N_2$; and $M_1$ is assigned to $N_4$, (d) allocation and PN schedules for Experiment IV: $M_4$, $M_5$, $M_6$, and $M_7$ are assigned to $N_1$; $M_8$, $M_9$, and $M_{10}$ are assigned to $N_2$; $M_2$ and $M_3$ are assigned to $N_3$. 
**List of Symbols**

- $P_{ND}$: the probability of no dynamic failure, i.e., the probability of all tasks making their deadlines.
- $P_{NDi}$: the probability that all tasks within a planning cycle are completed before their deadlines.
- $P_{NDD}$: the probability that all PNs are operational during the execution of modules assigned to them, and the communication links between communicating PNs are operational for all the intermodule communications that use these links.
- $N_f$: the total number of periodic tasks in the system.
- $t_i$: the invocation time of a periodic task $T_i$.
- $T_i$: the period of a periodic task $T_i$.
- $d_i$: the deadline of a periodic task $T_i$.
- $L$: the planning cycle of a set of periodic tasks. It is computed as the least common multiple of $\{p_i: i = 1, 2, \ldots, N_f\}$.
- $M_i \rightarrow M_j$: the precedence constraint imposed on modules $M_i$ and $M_j$, meaning that the completion of $M_i$ enables $M_j$ to be ready for execution.
- $c_i$: the execution time of a module $M_i$.
- $N$: the number of modules to be allocated within a planning cycle.
- $K$: the number of PNs available for module allocation.
- $x$: a module allocation where $x_k = 1$ if module $M_k$ is assigned to PN $N_k$.
- $TG$: the task flow graph representing the task system.
- $TG(x)$: the set of modules which are already allocated under the allocation $x$. $TG(x) = TG$ if $x$ is a complete allocation.
- $AN$: the set of active nodes in the search tree which needs to be considered for node expansion in the next stage.
- $x_{opt}$: an optimal allocation.
- $P^*_{ND}$: the objective function value achieved by $x_{opt}$.
- $P_{ND}(x)$: the value by which the objective function, $P_{ND}$, of all child nodes expanded from the allocation $x$ is upper-bounded.

**Notation Used in Sections 4-7**

- $p_{\ell}(T_k | x)$: the probability that a task $T_k$ is completed before its deadline under allocation $x$.
- $TG_\ell$: a component task graph of $TG$.
- $\{TG_\ell\}$: the set of component task graphs of $TG$.
- $p_\ell$: the probability that $TG_\ell$ is represented by $TG_\ell$.
- $TG_\ell(x)$: the set of modules $\in TG_\ell$ which are allocated under $x$.
- $S_k(x)$: the set of modules which are assigned to $N_k$ under allocation $x$, i.e., $S_k(x) = \{M_j: x_k = 1\}$.
- $r_i$: the release time of module $M_i$ which can be interpreted as the earliest time at which $M_i$ can start its execution.
- $LC_i$: the latest completion time of $M_i$, $M_i$ must be completed before $LC_i$ to ensure all tasks to meet their deadlines.
- $D_i$: the critical time of $M_i$, $M_i$ must be completed before $D_i$ to ensure that the corresponding task (to which $M_i$ belongs) will meet its deadline.
- $C_i$: the completion time of $M_i$ which is determined by MSA.
- $\hat{c}_i$: the modified execution time of $M_i$. $\hat{c}_i$ is used to include the effect of queuing $M_i$ on the release times of all those modules succeeding $M_i$.
- $f_j(C_i)$: the cost incurred by completing $M_j$ at time $C_i$.
- $com_j(x)$: the IMC time from $M_i$ to $M_j$ under allocation $x$.
- $d_{\ell}$: the IMC volume (measured in data units) from $M_i$ to $M_j$.
- $Y_{\ell}$: the nominal delay (measured in time units per data unit) between two PNs, $N_k$ and $N_{\ell}$.
- $B$: the minimal set of modules that are processed without any idle time from $r(B) = \min_{M_i \in B} r_i$ until $c(B) = r(B) + e(B)$, where $e(B) = \sum_{M_i \in B} e_i$.
- $b$: the number of blocks in $S_g(x)$.
- $\hat{d}_{\ell}$: the outdegree of $M_i$ within a block under consideration.
- $\hat{B}_i$: a subblock of $B - \{M_m\}$, where $1 \leq i \leq \hat{b}$, $\hat{b}$ is the number of subblocks in $B - \{M_m\}$, and $M_m$ is the module scheduled to be executed if no other modules in $B$ are waiting.
- $q_i$: the looping-back probability of the loop $L_i$.
- $n_{1,\ell}$: the maximum loop count of the loop $L_i$.
- $\hat{q}_{b,\ell}$: the branching probability of the $\ell$th branch of an OR-subgraph, $O_{\ell}$.
- $n_{O_{\ell}}$: the number of branches in the OR-subgraph $O_{\ell}$.
- $P_{n_{O_{\ell}}}(T_k | TG_{\ell}(x))$: the probability that a task $T_k$ is completed before its deadline under allocation $x$ for a given component task graph $TG_{\ell}$.
- $\hat{T}_\ell$: $\{M_j: M_j \in T_k \cap TG_{\ell}, d_{\ell} = 0 \text{ with respect to } T_k \cap TG_{\ell}\}$: the set of modules without any successor in $T_k \cap TG_{\ell}$.
- $LP$: the set of modules which are contained in loops.
- $OR$: the set of modules which are on branches of OR-subgraphs.
- $\lambda_c$: the constant exponential failure rate of $N_k$.
- $\hat{\lambda}_{mn}$: the constant exponential failure rate of link $\ell_{mn}$. We assume that $\lambda_s$ and $\hat{\lambda}_{mn}$ are statistically independent of one another.
- $\ell_{mn}$: the nominal delay (measured in time units per data unit) of link $\ell_{mn}$.
- $n(k, \ell)$: the number of edge-disjoint paths from $N_k$ to $N_{\ell}$.
- $I(n, n, k, \ell)$: the indicator variable such that $I(n, n, k, \ell) = 1$ if $\ell_{mn}$ lies on one of the $n(k, \ell)$ edge-disjoint paths from $N_k$ to $N_{\ell}$.
- $R_m(i, j, n, x)$: the probability that link $\ell_{mn}$ is operational during $n$ occurrences of IMC between $M_i$ and $M_j$ under allocation $x$.
- $R_{ym}(x)$: the probability that all PNs are operational during the execution of modules assigned to them under allocation $x$.
- $R_{sig}(x)$: the probability that all links are operational for performing all the IMCs that use them under allocation $x$.
- $LC_i$: a pessimistic estimate of $LC_i$ used in the branching process.
- $LC_i^o$: an optimistic estimate of $LC_i$ used in the branching process.
- $r_i^c$: an optimistic estimate of $r_i$ used in the branching process.
- $PN$: the set of PNs who need to reschedule the modules assigned to them because of the addition of $M_i \rightarrow N_k$ to a partial allocation.
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