SECOND ORDER NECESSARY CONDITIONS FOR OPTIMAL CONTROL PROBLEMS WITH CONTROL AND INTEGRAL CONSTRAINTS

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ABSTRACT

An optimal control problem containing restrictions on control variables and integral equality and inequality constraints is considered. A relatively simple and self-contained method for deriving new second order necessary conditions for such problems is presented. The new conditions derived here generalize some of the results reported in [3], [6], [9]. The method developed is quite general and is also applicable to optimal control problems including state or mixed state-control inequality constraints and to optimal control problems governed by partial differential equations.

1. INTRODUCTION

In a recent paper, Warga [1] presented a simple and self-contained derivation of new second order necessary conditions for an abstract optimization problem containing restrictions in the form of finitely many equality constraints and in the form of possibly infinite-dimensional inclusions in closed convex sets. His elegant proof is based on the separation of convex sets and on a suitable fixed point theorem. Unlike somewhat similar necessary conditions obtained by Bernstein [2] and Bernstein and Gilbert [3], the Lagrange-type multipliers \( Y \) of Warga [1] may be common to all elements of a certain set \( Y \) of critical variations (see Section 4 below) and need not be chosen separately for each critical variation. Furthermore, the set \( C \) of \( [1] \) defining the infinite dimensional constraints may be an arbitrary closed convex set and not necessarily a cone. These appear to be important improvements over the results reported in [2], [3], [4]. See [5] for a survey of higher order necessary conditions in unrestricted optimal control problems. For related work (with different approaches) on restricted problems, see [2], [3], [4], [6], [7], [8] and references therein.

The main goal of this paper is to present a self-contained and relatively simple approach to second order conditions in optimal control problems (governed by ordinary differential equations) including constraints on the control variables, and integral equality and inequality constraints. The second order conditions that we obtain here generalize Theorem 3.1 in [2] and also the results in [6]. The theory developed can also be extended with little modification to relaxed controls (although we do not consider relaxed controls here) and, therefore, generalize some of the results of [3] and [4]. In a forthcoming paper we shall apply the basic ideas we develop here to optimal control problems with state or mixed state-control inequality constraints. The approach makes use of the dependence of solutions of differential equations on parameters. In Section 6 proofs are completed, and the paper concludes with Section 7.

2. STATEMENT OF THE PROBLEM

Let \( I := [t_1, t_2] \) be a fixed, closed, and bounded interval of the real numbers \( R \) and \( S \subseteq R^r \). Let \( M(I,S) \), \( L_w(I,S) \), and \( AC(I,S) \) denote, respectively, the linear spaces of all functions \( F : I \rightarrow S \) which are measurable, measurable and essentially bounded, and absolutely continuous. For open sets \( X \subseteq R^n \) and \( U \subseteq R^m \), define

\[
X := AC(I,X), \quad U \subseteq M(I,U).
\]

\[ f : I \times X \times U \rightarrow R^r, \]

\[ \phi_i, \psi_j : I \times X \times U \rightarrow R, \quad i = 0, 1, \ldots, M; \quad j = 1, \ldots, N, \]

\[ \phi := (\phi_0, \ldots, \phi_M), \quad \psi := (\psi_1, \ldots, \psi_N), \]

\[ \psi := (\psi_0, \phi_1, \ldots, \phi_M, \psi_1, \ldots, \psi_N). \]

The optimal control problem is to find a pair \((x(t), u(t))\) in \( X \times U \) which minimizes the functional

\[ J(x, u) := \int_{t_1}^{t_2} \phi(t, x(t), u(t)) dt \]

subject to

\[ \frac{dx(t)}{dt} = f(t, x(t), u(t)), \quad \text{a.a. } t \in I, \quad x(t_1) = x_1, \]

\[ \int_{t_1}^{t_2} \phi_i(t, x(t), u(t)) dt = 0, \quad i = 1, \ldots, M, \]

\[ \int_{t_1}^{t_2} \psi_j(t, x(t), u(t)) dt \leq 0, \quad j = 1, \ldots, N. \]

3. NOTATION AND ASSUMPTIONS

It is necessary to introduce the following notation and assumptions concerning the functions defining the optimal control problem.

Let \( A \) be an open subset of \( R^r \). The first derivative of a \( C^1 \) map \( F : A \rightarrow R^r \) at \( \alpha \in A \) is defined to be the linear map

\[ F' (\alpha) : R^r \rightarrow R^r, \quad y = F' (\alpha) x, \]
where $F'(\overline{a})$ is identified with the Jacobian of $F$ at $\overline{a}$. If $F$ is $C^2$ on $A$, then the second derivative of $F$ at $\overline{a}$ is defined to be the bilinear map

$$F''(\overline{a}): \mathbb{R}^r \times \mathbb{R}^r \rightarrow \mathbb{R}^s, \quad y = z^T F''(\overline{a}) z.$$

Here $T$ denotes the transpose and $F''(\overline{a})$ is the Hessian matrix of $F$ at $\overline{a}$. If $F$ is $C^2$ on $A$, then the second derivative of $F$ at $Z$ is defined to be the bilinear map

$$F''(\overline{a})(Z) = z^T F''(\overline{a}) z.$$

Now suppose that $F: A_1 \times A_2 \rightarrow \mathbb{R}^s$ is $C^1$, where $A_1 \subseteq \mathbb{R}^{r_1}$, $A_2 \subseteq \mathbb{R}^{r_2}$ are open, and let $\overline{a} = (\overline{a}_1, \overline{a}_2) \in A_1$. The first partial derivative of $F$ with respect to $a_1$ at $\overline{a}$, denoted by $F_{a_1}(\overline{a})$, is the first derivative of the map $F : A_1 \rightarrow \mathbb{R}^s$, where $F(a_1) = F(a_1, \overline{a}_2)$. $F_{a_1}(\overline{a})$ is defined in the obvious manner. If $F$ is $C^2$, the second partial derivative,

$$F_{a_1 a_2}(\overline{a}): \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \rightarrow \mathbb{R}^s,$$  

is a bilinear map identified with $r_1 \times r_2$ matrices of the second partial derivatives of $F_{a_1}$.

**Assumptions**

A1. For each $t \in I$, the functions

\[
(x, u) \rightarrow f(t, x, u) : X \times U \rightarrow \mathbb{R}^s,
\]

\[
(x, u) \rightarrow \Phi(t, x, u) : X \times U \rightarrow \mathbb{R}^{1+M},
\]

\[
(x, u) \rightarrow \psi(t, x, u) : X \times U \rightarrow \mathbb{R}^N
\]

are all $C^2$.

A2. For each $(x, u) \in X \times U$, the functions

\[
t \rightarrow f(t, x, u) : I \rightarrow \mathbb{R}^s,
\]

\[
t \rightarrow \Phi(t, x, u) : I \rightarrow \mathbb{R}^{1+M},
\]

\[
t \rightarrow \psi(t, x, u) : I \rightarrow \mathbb{R}^N
\]

are all measurable.

A3. There exists a nonnegative integrable function $k : I \rightarrow R$ such that for all $(t, x, u) \in I \times X \times U$,

\[
|f'| + |f_x| + |f_u| + |f_{xx}| + |f_{vu}| + |f_{uu}| \leq k,
\]

\[
|\Phi| + |\Phi_z| + |\Phi_u| + |\Phi_{xz}| + |\Phi_{zu}| + |\Phi_{uu}| \leq k,
\]

\[
|\psi| + |\psi_z| + |\psi_u| + |\psi_{xz}| + |\psi_{zu}| + |\psi_{uu}| \leq k.
\]

A4. $U$ is convex in $\mathbb{R}^m$.

Throughout this paper, $(\bar{x}(\cdot), \bar{u}(\cdot)) \in \bar{X} \times \bar{U}$ denotes an optimal pair, i.e., a solution to the optimal control problem (1) - (4). The evaluation of functions on $(\bar{x}(\cdot), \bar{u}(\cdot))$ will be represented by a superbar, e.g., $\bar{f}_x(t) := f_x(t, \bar{x}(t), \bar{u}(t))$.

**4. MAIN RESULTS**

In this section we state main results of the paper and give the proofs in the following sections.

We begin by defining the sets $\bar{X}$ and $\bar{U}$ of, respectively, state and control variations.

(i) $\bar{X} := AC(I, \mathbb{R}^s)$,

(ii) $\bar{U} \subseteq L^\infty(I, \mathbb{R}^m)$,

(iii) $\bar{U} \subseteq \mathbb{U} - \bar{U} := \{v(t) \mid v(t) = u(t) - \bar{u}(t), u(t) \in U\}$,

(iv) $0 \in \bar{U}$.

The set of critical variations is defined as one of the following two sets

\[
Y_\alpha := \{y(t), v(t) \in \bar{X} \times \bar{U} \mid (y(t), v(t)) \text{ satisfies } (\alpha)\},
\]

\[
Y_\alpha^\beta := \{y(t), v(t) \in \bar{X} \times \bar{U} \mid (y(t), v(t)) \text{ satisfies both } (\alpha) \text{ and } (\beta)\},
\]

where

\[
(\alpha) \int_{t_1}^{t_2} \left[ \bar{\phi}_{0z}(t)y(t) + \bar{\phi}_{0u}(t)v(t) \right] dt \leq 0,
\]

\[
\int_{t_1}^{t_2} \left[ \bar{\phi}_z(t)y(t) + \bar{\phi}_u(t)v(t) \right] dt = 0,
\]

\[
\int_{t_1}^{t_2} \left[ \bar{\psi}_{jz}(t)y(t) + \bar{\psi}_{ju}(t)v(t) \right] dt \leq 0, \quad j = 1, \ldots, N;
\]

\[
\frac{dy(t)}{dt} = f_x(t)y(t) + f_u(t)v(t), \text{ a.a. } t \in I, \ y(t_1) = 0.
\]

(\beta) For $(y(t), v(t))$ and $(\eta(t), w(t))$ in $Y_\alpha$ with $v(t) \neq w(t)$,

\[
\int_{t_1}^{t_2} \left[ \bar{\phi}_{zz}(t)(y(t), \eta(t)) + \bar{\phi}_{zu}(t)(v(t), w(t)) \right] dt \leq 0,
\]

\[
\int_{t_1}^{t_2} \left[ \bar{\psi}_{jz}(t)(y(t), \eta(t)) + \bar{\psi}_{ju}(t)(v(t), w(t)) \right] dt \leq 0, \quad j = 1, \ldots, N.
\]

**Remark 4.1.** Recall that for unrestricted problems, that is for $M = 0, N = 0$, an optimal control $\bar{u}(t)$ is said to be singular in the sense of Pontryagin's maximum principle, if there is a nonempty subset $\bar{U}$ of $U - \bar{U}$ (with $U$ convex) such that (see [5],[9])

\[
H_\alpha(t, \bar{x}(t), \bar{u}(t), \lambda(t))v(t) = 0 \text{ for all } v(t) \in \bar{U}.
\]

Here $H = \phi_0 + \lambda^T f$, $d\lambda(t)/dt = -H_x(t, \bar{x}(t), \bar{u}(t), \lambda(t))$, and $\lambda(t_2) = 0$. Condition (9) is equivalent to saying that $\int_{t_1}^{t_2} \phi_{0z}(t, \bar{x}(t), \bar{u}(t))y(t) dt = 0$, where
\( y(t) \) is the state variation defined in (7). Thus the above definition of critical variation can be regarded as an extension of this definition of singular controls to the constrained optimal control problems.

**Remark 4.2.** Condition (8) is essential in proving satisfactory second order optimality conditions in restricted optimization problems if it is desired to have a Lagrange-type multiplier \( l \) independent of critical variations (that is, \( l \) common to all elements of the set \( Y_{\alpha \phi} \)). See [1]. However, in the absence of (8), one can derive somewhat less satisfactory conditions in which the multipliers \( l \) have to be chosen separately for each critical variation (see Theorem 4.2 below and [3]).

In what follows, the following definitions will be used.

In what follows, the following definitions will be used.

\[ l := (l_0, l_1, l_2) := (l_0, l_1, l_{21}, \ldots, l_{2N}) \in \mathbb{R}^{1+M+N}, \]

\[ H(t, x, u, \lambda) := l^T \Psi + \lambda^T f := l_0 \phi_0 + l_1 \phi + l_2 \phi + \lambda^T f \]

\[ J_\phi := \left\{ j \in \{1, \ldots, N\} \left| \int_{t_1}^{t_2} \tilde{\psi}(t) \, dt = 0 \right. \right\} \]

We now state our main results.

**Theorem 4.1.** If \((\bar{F}(\cdot), \bar{N}(\cdot))\) solves the problem (1) - (4) under the assumptions A1 - A4, then there exist \( l \in \mathbb{R}^{1+M+N} \) and \( \lambda(\cdot) \in \tilde{X} \), such that

\[ l \neq 0, \quad l_0 \geq 0, \]

\[ l_{2j} \geq 0, \quad j \in \{1, \ldots, N\}, \]

\[ l_{2j} = 0, \quad j \in \{1, \ldots, N\} \setminus J_\phi, \]

\[ \lambda(t) = \int_{t_1}^{t_2} \left[ l^T \tilde{\psi}_t(t) + \lambda^T(t) \bar{F}_t(t) \right] \, dt, \]

\[ \int_{t_1}^{t_2} \bar{H}_v(t, \lambda(t))v(t) \, dt \geq 0, \quad \text{for all } v(\cdot) \in \bar{U}, \]

\[ \int_{t_1}^{t_2} \left[ \bar{H}_{ss}(t, \lambda(t))(y(t))^2 + 2\bar{H}_{sv}(t, \lambda(t))(y(t), v(t)) + \bar{H}_{vv}(t, \lambda(t))(v(t))^2 \right] \, dt \geq 0, \quad \text{for all } (y(t), v(\cdot)) \in Y_{\alpha \phi}. \]

**Remark 4.3.** Note that condition (16) holds for all elements \((y(\cdot), v(\cdot)) \in Y_{\alpha \phi}\). That is, the theorem asserts that a Lagrange-type multiplier \( l \) and the corresponding \( \lambda \) exist such that (16) is satisfied for all elements of \( Y_{\alpha \phi} \).

**Theorem 4.2.** Under the same conditions of Theorem 4.1, for each \((y(\cdot), v(\cdot)) \in Y_{\alpha \phi}\) there exist \( l \in \mathbb{R}^{1+M+N} \) and \( \lambda(\cdot) \in \tilde{X} \) such that (11) - (16) are satisfied and

\[ \int_{t_1}^{t_2} \left[ \bar{H}_{ss}(t, \lambda(t))(y(t))^2 + 2\bar{H}_{sv}(t, \lambda(t))(y(t), v(t)) + \bar{H}_{vv}(t, \lambda(t))(v(t))^2 \right] \, dt \geq 0. \]

**Remark 4.4.** Theorem 4.1 corresponds to Theorem A in [1] for abstract optimization problems. It generalizes Theorem 3.1 in [3] and thus the result in [6]. Theorem 4.2 is less satisfactory in the sense that the second order optimality condition (10) may not hold for all \((y(\cdot), v(\cdot)) \in Y_{\alpha \phi}\). In other words, the Lagrange-type multiplier \( l \) may not be common to all elements of the set of critical variations. We will prove both Theorems 4.1 and 4.2 by the same argument (although the proof is much simpler for Theorem 4.2).

For additional remarks and elaborations on Theorem 4.1, we refer the reader to [3].

**5. PARAMETER DEPENDENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS**

In this section we consider a suitable perturbation of the differential equation (2) and study the dependence of the solution of the perturbed equation on the parameters defining the perturbation. Materials in this section are a basis to the proof of Theorems 4.1 and 4.2.

Let \( \epsilon, | \epsilon | < 1 \), be a small parameter and let \( v(\cdot), w(\cdot) \in \bar{U} \) be bounded measurable functions. Clearly, for sufficiently small \( \epsilon \), the function

\[ u(\cdot) := \bar{u} + \epsilon v + \frac{\epsilon^2}{2} w \]

belongs to the open set \( U \). Consider now the initial value problem:

\[ \frac{dx(t)}{dt} = f(t, x(t), u(t)), \quad \text{a.a. } t \in I, \quad x(t_1) = z_1. \]

For \( \epsilon = 0 \) we have \( z(t) = \bar{z}(t), \quad \text{a.a. } t \in I \) by the uniqueness of solutions of ordinary differential equations. Let \( y(\cdot) \) be the solution of the linear initial-value problem:

\[ Ly(t) = \bar{f}_s(t) v(t), \quad y(t_1) = 0. \]

where

\[ L := \frac{dy}{dt} - \bar{f}_s y. \]

The derivative \( \frac{\partial x(t)}{\partial \epsilon} \) exists (see, e.g., [10], Theorem II.4.9, p. 165) and for \( \epsilon = 0 \) we have

\[ \left. \frac{\partial x(t)}{\partial \epsilon} \right|_{\epsilon=0} = y(t), \quad \text{a.a. } t \in I. \]

The second derivative of \( x(t) \) at \( \epsilon = 0 \) exists by the same argument and thus setting

\[ z(t) := \left. \frac{\partial^2 x(t)}{\partial \epsilon^2} \right|_{\epsilon=0} \]

we have for a.a. \( t \in I, \)

\[ Lz(t) = \bar{f}_s(t) w(t) + \bar{f}_{ss}(t)(y(t))^2 + 2\bar{f}_{sv}(t)(y(t), v(t)) + \bar{f}_{vv}(t)(v(t))^2, \]

\[ z(t_1) = 0. \]

Here the linear operator \( L \) and the function \( y \) are as defined in (19).

Next let \( \epsilon \geq 0, \quad c_{ij} \geq 0, \quad \text{and } d_{i}, \quad d_{i} \geq 0 \) be nonnegative real numbers satisfying

\[ \sum_{i=1}^{k} c_{ij} = 1 \quad \text{for each } i = 0,1, \ldots, M. \]
The sum \( \sum_{r=1}^{i} d_r = 1 \) for each \( i = 0, 1, \ldots, M \),

\[
e^k \sum_{j=1}^{k} \left( \sum_{i=0}^{M} c_{ij} \right)^{1/2} + \frac{1}{2} \cdot \epsilon^2 \sum_{r=1}^{i} \left( \sum_{i=0}^{M} d_r \right) \leq 1,
\]

where \( k \) and \( l \) are arbitrary positive integers. Also let

\[
\theta := (\theta_0, \theta_1, \ldots, \theta_M), \quad \theta_i \geq 0, \quad |\theta| := \sum_{i=0}^{M} \theta_i = 1.
\]

Finally, define

\[
u_s(t) := \tilde{u}(t) + \epsilon v(t) + \frac{1}{2} \epsilon^2 w(t)
\]

\[
= \tilde{u}(t) + \epsilon \sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^{1/2} v_j(t) + \frac{1}{2} \epsilon^2 \sum_{r=1}^{i} \left( \sum_{i=0}^{M} d_r \theta_i \right) w_r(t),
\]

where \( v_j(\cdot) \) and \( w_j(\cdot) \) belong to \( U \). Obviously, for \( 0 \leq \epsilon \ll 1 \), \( u(\cdot) \) takes values in the set \( U \). Consider now the initial-value problem (18) with \( u(\cdot) \) defined in (21):

\[
\frac{dx(t)}{dt} = f(t, x(t), u(t)), \quad \text{a.a. } t \in I, \quad x(t_1) = z(t_1)
\]

Lemma 5.1. The solution \( x(\cdot) \) of the initial-value problem (22) is of the form

\[
x_s(t) = \tilde{u}(t) + \epsilon \sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^{1/2} y_j(t) + \frac{1}{2} \epsilon^2 \left( \sum_{i=0}^{M} d_i \theta_i \right) z_r(t) + o(\epsilon^2),
\]

where \( o(\epsilon^2) \to 0 \) uniformly in \( t \) as \( \epsilon \to 0+ \) and \( y_j(\cdot) \) and \( z_r(\cdot) \) are solutions of

\[
L y_j(t) = \tilde{u}(t) v_j(t), \quad \text{a.a. } t \in I, \quad y_j(t_1) = 0, \quad j = 1, \ldots, k,
\]

\[
L z_r(t) = \tilde{u}(t) w_r(t) + \tilde{z}_{zu}(t)(y(t))^2 + 2 \tilde{z}_{zu}(t)(y(t), v(t)) + \tilde{z}_{uz}(t)(v(t))^2,
\]

\[
\text{a.a. } t \in I, \quad z_r(t_1) = 0, \quad r = 1, \ldots, l,
\]

where

\[
y(t) := \sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^{1/2} y_j(t), \quad \text{a.a. } t \in I.
\]

Proof. Since the first and second derivatives of \( x(\cdot) \) with respect to \( t \) exist, we can write

\[
x_s(t) = \tilde{u}(t) + \epsilon y(t) + \frac{1}{2} \epsilon^2 z(t) + o(\epsilon^2).
\]

\[
L z = \tilde{f}_u w + \tilde{f}_{zu} y^2 + 2 \tilde{f}_{zu}(y, v) + \tilde{f}_{uz} v^2,
\]

\[
z(t_1) = 0, \quad w \text{ as defined in (21)}.
\]

We have to specify the forms of \( y(\cdot) \) and \( z(\cdot) \) in these relations. It follows from (23) that

\[
L \sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^{1/2} y_j(t) = \tilde{f}_u(t) \sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^{1/2} v_j(t), \quad \text{a.a. } t \in I,
\]

\[
\sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^{1/2} y_j(t_1) = 0.
\]

Comparing (23) and (27) and taking into account the uniqueness of solution, we see that

\[
y(t) = \sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^{1/2} y_j(t).
\]

Similarly, from (24) and definitions of \( c_{ij} \) and \( d_r \), we have

\[
L \sum_{r=1}^{l} \left( \sum_{i=0}^{M} \theta_i d_{ir} \right) z_r(t) = \tilde{f}_u(t) \sum_{r=1}^{l} \left( \sum_{i=0}^{M} \theta_i d_{ir} \right) w_r(t) + \tilde{f}_{zu}(t)(y(t))^2 + 2 \tilde{f}_{zu}(t)(y(t), v(t)) + \tilde{f}_{uz}(t)(v(t))^2,
\]

\[
\text{a.a. } t \in I, \quad \sum_{r=1}^{l} \left( \sum_{i=0}^{M} \theta_i d_{ir} \right) z_r(t_1) = 0.
\]

Comparing (28) and (29), we see by the uniqueness of solution that \( z(\cdot) \) has the form asserted in the Lemma. This completes the proof.

Lemma 5.2.

\[
\int_{t_1}^{t_2} \Phi(t, x_s(t), u_s(t)) dt = \int_{t_1}^{t_2} \Phi(t) dt + \frac{1}{2} \epsilon^2 \theta + o(\epsilon^2 \theta) + (a_0, o)
\]

where \( o(\epsilon^2 \theta) \to 0 \) uniformly in \( t \) as \( \epsilon \to 0+ \). \( H \) is an \((M + 1) \times (M + 1)\) matrix whose columns are defined by the \((M + 1)\) vectors

\[
\int_{t_1}^{t_2} \Phi_{zu}(t) \sum_{j=1}^{k} c_{ij}(y_j(t))^2 + \Phi_{uz}(t) \sum_{j=1}^{k} d_{ir} z_r(t)
\]

\[
+ \Phi_{zu}(t) \sum_{j=1}^{k} c_{ij}(y_j(t))^2 + \Phi_{uz}(t) \sum_{r=1}^{l} d_{ir} w_r(t) \int_{t_1}^{t_2} dt,
\]

\[
L z = \tilde{f}_u w + \tilde{f}_{zu} y^2 + 2 \tilde{f}_{zu}(y, v) + \tilde{f}_{uz} v^2,
\]

\[
z(t_1) = 0, \quad w \text{ as defined in (21)}.
\]
and where \( \tilde{a}_0 \) is the nonpositive number

\[
\tilde{a}_0 := \epsilon \int_{t_1}^{t_2} \left[ \sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^2 \mathcal{F}_{yj}(t) \right] dt + \frac{1}{2} \epsilon^2 \int_{t_1}^{t_2} \left[ \sum_{j=1}^{k} \left( \sum_{i=0}^{M} \theta_i c_{ij} \right)^2 \mathcal{F}_{vj}(t) \right] dt
\]

where \( \mathcal{F}_{yj}(t) \) and \( \mathcal{F}_{vj}(t) \) are the nonpositive numbers.

Proof. Write the first three terms of the Taylor series expansion of \( \Phi \) about \( \epsilon = 0 \), use Lemma 5.1 and the fact that \( (y_j(\cdot),v_j(\cdot)) \in \mathcal{Y}_{o\beta} \).

6. COMPLETION OF PROOFS

We shall use the following lemma on separation of convex sets, which is a special case of Lemma V.2.1, p.299 of [10].

Lemma 6.1. Let \( C \) be a convex subset of \( \mathbb{R} \times \mathbb{R}^M \times \mathbb{R}^N \), \( K \) an open convex subset of \( \mathbb{R}^N \), \( \mathcal{O} \subset \mathcal{C} \), \( \mathcal{O} \subset \mathcal{K} \). Then either either exists \( l := (l_0,l_1,l_2) \in [0,\infty) \times \mathbb{R}^M \times \mathbb{R}^N \) such that

\[
i \neq 0, \\
l_0 \geq 0, l_1 c_1 + l_2 c_2 \geq 0 \quad \forall \; c := (c_0, c_1, c_2) \in \mathcal{C}, \\
l_0 k \leq 0 \quad \forall \; k \in \mathcal{K},
\]

or there exist points \( \xi^i := (\xi^i_0, \xi^i_1, \xi^i_2) \in \mathcal{C} \) and numbers \( \beta_i > 0, i = 0, 1, \ldots, M \) such that \( \sum_{i=0}^{M} \beta_i = 1 \), \( \xi^i \in \mathcal{K} \), the set \( \{ (\xi^0_0, \xi^1_0), \ldots, (\xi^M_0, \xi^M_1) \} \) is linearly independent, \( \xi^i_0 < 0, i = 0, 1, \ldots, M \), and \( \sum_{i=0}^{M} \beta_i \xi^i_1 = 0 \).

Proof of Theorem 4.1. Define the sets

\[
K := \{ k \in \mathbb{R}^N \mid k_j < 0 \text{ for } j \in J \}
\]

\[
C := \{ c \in \mathbb{R}^1 \times \mathbb{R}^{M+N} \mid (y(\cdot),v(\cdot)) \in \mathcal{Y}_{o\beta}, \text{ (z(\cdot),w(\cdot)) satisfy (20) } \}
\]

\[
l \neq 0, l_0 \geq 0, l_2 k \leq 0 \quad \forall \; k \in \mathcal{K}
\]

(29)

(30)

where \( \alpha \) denotes the convex hull. Clearly, \( K \) and \( C \) are convex, \( \mathcal{O} \subset \mathcal{C} \) and \( \mathcal{K} \) is open. Thus by Lemma 5.1, either there exists \( l := (l_0,l_1,l_2) \in [0,\infty) \times \mathbb{R}^M \times \mathbb{R}^N \) satisfying

Remark 5.1. Similar expansion holds for \( \psi := (\psi_1, \ldots, \psi_N) \).

\[
l \neq 0, \; l_0 \geq 0, \; \forall \; c \in \mathcal{C}, \quad \text{and } \; l_2 k \leq 0 \quad \forall \; k \in \mathcal{K}.
\]

For all \( (y(\cdot),v(\cdot)) \in \mathcal{Y}_{o\beta} \) and all \( (z(\cdot),w(\cdot)) \) satisfying (20).

We prove that the second alternative cannot hold for the above choices of \( K \) and \( C \). So assume, on the contrary, that the second alternative holds. Thus there are nonnegative numbers \( c_{ij}, \sum_{i=0}^{M} c_{ij} = 1 \) for \( i = 0, 1, \ldots, M \) and \( (y_j(\cdot),v_j(\cdot)) \in \mathcal{Y}_{o\beta} \), \( (z_j(\cdot),w_j(\cdot)) \) satisfying (21) such that

\[
\xi^i := (\xi^i_0, \xi^i_1, \xi^i_2) = \sum_{j=1}^{k} c_{ij} \int_{t_1}^{t_2} \left[ \mathcal{F}_{yj}(t)(y_j(t))^2 + 2\mathcal{F}_{v_j}(t)(y(t),v(t)) + \mathcal{F}_{w_j}(t)(v(t))^2 \right] dt.
\]

(30)

By Lemma 5.2 we have

Using \( c_{ij} \) and \( d_{ij} \), the functions \( v_j(\cdot) \) and \( w_j(\cdot) \) in (21) we define \( u_j(\cdot) \) as in (20) and find the corresponding \( z_j(\cdot) \) in (21).
\[ \int_{t_1}^{t_2} \mathcal{F}(t, x(t), u(t)) dt = \int_{t_1}^{t_2} \mathcal{F}(t) dt + \frac{1}{2} \varepsilon^2 H \theta + \alpha(\varepsilon^2 \theta) + \tilde{\alpha}(0), \]  

(32)

where now the \((1 + M) \times (1 + M)\) matrix \(H\) with linearly independent columns \(\{\xi_0, \xi_1\}\) defined in (31), \(i = 0, 1, \ldots, M\), is nonsingular. Let

\[ \beta_{\min} := \min \{ \beta_0, \beta_1, \ldots, \beta_M \} \]

and observe that there exists \(\tilde{\varepsilon} > 0\) such that

\[ 2H^{-1} \alpha(\varepsilon^2 \theta) \leq \frac{\varepsilon}{3} \beta_{\min} \quad \text{for} \quad |\theta| = 1 \quad \text{and} \quad 0 \leq \varepsilon < \tilde{\varepsilon} \]  

(33)

Choose \(\tilde{\varepsilon}\) sufficiently small so that

\[ \tilde{\varepsilon} \leq \frac{2}{3} \tilde{\varepsilon}, \quad S := \left\{ s = (s_0, s_1, \ldots, s_M) \in \mathbb{R}^{1+M} \middle| \begin{array}{l}
|s_i - \gamma \beta_i| \leq \frac{1}{2} \gamma \beta_{\min}, \quad 0 \leq i \leq M \\
\end{array} \right\} \]

The set \(S\) is clearly compact and convex and

\[ s \in S \text{ implies } 0 \leq \frac{1}{2} \gamma \beta_{\min} \leq s_i \text{ and } |s| \leq \tilde{\varepsilon}. \]  

(34)

Furthermore, the mapping

\[ s \rightarrow \gamma \beta - \frac{2}{|s|} H^{-1} \alpha \left( \frac{s}{|s|} \right) : S \rightarrow \mathbb{R}^{1+M}, \]

where \(\beta := (\beta_0, \beta_1, \ldots, B_M)\) is continuous and it easily follows from (33)-(34) that it maps \(S\) into \(S\). Therefore, it has a fixed point \(\hat{s} \in S\). Let us write \(s = \varepsilon \theta, \hat{s} = \tilde{s} \theta, |\theta| = 1\). Thus we have shown that for some \(\varepsilon > 0\) and \(\varepsilon \theta = \hat{s}, |\theta| = 1, \) thus we have shown that for some \(\varepsilon > 0\) and \(\varepsilon \theta = \hat{s}, |\theta| = 1, \)

\[ \alpha(\varepsilon^2 \theta) = \frac{1}{2} \gamma H \beta - \frac{1}{2} \varepsilon^2 H \theta \]

and so, defining \(x(\cdot)\) and \(\hat{a}_0\) the same way as \(x(\cdot)\) and \(a_0\) but with \(\varepsilon, \theta\) replacing \(\varepsilon, \theta\), it follows from (32) that

\[ \int_{t_1}^{t_2} \Phi(t, x(t), u(t)) dt = \int_{t_1}^{t_2} \tilde{\Phi}(t) dt + \frac{1}{2} \varepsilon \gamma H \beta + (\hat{a}_0, 0), \]

from which we deduce

\[ \int_{t_1}^{t_2} \tilde{\Phi}(t, x(t), u(t)) dt = \int_{t_1}^{t_2} \tilde{\Phi}(t) dt + \frac{1}{2} \varepsilon \gamma M \sum_{i=0}^{M} \beta_i \xi_i^T + \hat{a}_0 \]

\[ < \int_{t_1}^{t_2} \tilde{\Phi}(t) dt + \hat{a}_0 \leq \int_{t_1}^{t_2} \tilde{\Phi}(t) dt, \]

\[ \int_{t_1}^{t_2} \Phi(t, x(t), u(t)) dt = \int_{t_1}^{t_2} \Phi(t) dt + \frac{1}{2} \varepsilon \gamma M \sum_{i=0}^{M} \beta_i \xi_i^T \]

\[ = \int_{t_1}^{t_2} \Phi(t) dt = 0. \]

We need to show that the pair \((x_0(\cdot), u_0(\cdot))\) satisfies the inequality constraints (4). An expression similar to the one in (32) yields

\[ \int_{t_1}^{t_2} \tilde{\psi}(t, x_0(t), u_0(t)) dt = \int_{t_1}^{t_2} \tilde{\psi}(t) dt + \frac{1}{2} \varepsilon \gamma M \sum_{i=0}^{M} \beta_i \xi_i^T \]

\[ + \frac{1}{2} \varepsilon \gamma M \sum_{i=0}^{M} \beta_i \xi_i^T + \tilde{\alpha}(0), \]

\[ = \int_{t_1}^{t_2} \tilde{\psi}(t) dt + \frac{1}{2} \varepsilon \gamma M \sum_{i=0}^{M} \beta_i \xi_i^T + \tilde{\alpha}(0). \]

(35)

where \(\tilde{a}_2 := (\tilde{a}_2_1, \ldots, \tilde{a}_2_N)\), \(\tilde{a}_2_j \leq 0\), is defined similar to \(a_0\) but with \(\psi\) replacing \(\phi_0\), and where \(\alpha(\varepsilon^2 \theta) \equiv 0\) as \(\varepsilon \rightarrow 0\) uniformly in \(\theta\). Noting that \(\xi_i \in K\), and \(a_2_j \leq 0, j = 1, \ldots, N\), we see that for sufficiently small \(\varepsilon\) (specifically \(\varepsilon = \tilde{\varepsilon}\)) each component of the right-hand side of (33) is nonpositive. In conclusion, we have proven that the pair \((x_0(\cdot), u_0(\cdot))\) is an admissible one at which the cost functional is less than the minimum value. This contradiction shows that the second alternative cannot hold. We now derive the optimality conditions (11)-(18) from (29) and (30).

Conditions (11)-(13) are just equivalent forms of (29). To derive (14)-(18), define the absolutely continuous function \(\lambda(\cdot)\) on \(I\) by

\[ \lambda(t) = \int_{t}^{t_2} \left[ t^T \mathcal{W}_z(t) + \lambda(t)^T \mathcal{F}(z(t)) \right] dt \]

(36)

Setting \(v = 0\) in (30) we have

\[ \int_{t_1}^{t_2} \left[ \mathcal{W}_z(t) z(t) + \mathcal{W}_u(t) u(t) \right] dt \geq 0 \]

(37)

for all \((z(\cdot), u(\cdot))\) satisfying (20). But notice that (20) with \(v = 0\) is equivalent to
\[
\frac{dz(t)}{dt} = \mathcal{I}_s(t)z(t) + \mathcal{J}_u(t)w(t), \quad z(t_1) = 0. \tag{38}
\]

Application of integration by parts to the first term of (37) yields
\[
\int_{t_1}^{t_2} T \mathcal{W}_s(t)z(t) dt = \int_{t_1}^{t_2} \left[ \int_{t_1}^{t_2} T \mathcal{W}_s(t) d\tau \right] \frac{dz(t)}{dt} dt \tag{39}
\]
Substituting from (38) into (39), using (38), simplifying, and finally substituting the result back into (37) yields (15).

To derive (16), we again use (30) but this time set \( w = 0 \), so that \( z \) now satisfies (see (20)).
\[
\frac{dz(t)}{dt} = \mathcal{I}_s(t)z(t) + \mathcal{J}_u(t)(y(t) + c\Phi \{ f \}) + \mathcal{J}_u(t)(y(t) + c\Phi \{ f \})^2 + 2\mathcal{J}_u(t)(y(t) + c\Phi \{ f \}) \tag{40}
\]
\[+ \mathcal{J}_u(t)(v(t)) \]
where \((y(t), v(t))\) has been defined in (19). Use the relation (39) with \( z \) as in (40) and repeat the above calculations to derive (16).

Proof of Theorem 4.2. The proof is almost exactly the same as that of Theorem 4.1. Here we define \( K \) as above but for each \((y(\cdot), v(\cdot))\) in \( Y_\alpha \), the set \( C \) is defined as
\[
C_{(y, v)} = \left\{ t_2 \int_{t_1}^{t_2} \left[ \alpha \mathcal{W}_s(t)(y(t))^2 + 2\alpha \mathcal{W}_u(t)(y(t), v(t)) + \mathcal{W}_u(t)(v(t))^2 + \mathcal{W}_u(t)z(t) + \mathcal{W}_u(t)w(t) \right] dt \right. \\
\left. \in \mathbb{R}^{1+M+N} \quad 0 < \alpha \leq 1, \text{ and } (z(\cdot), w(\cdot)) \text{ satisfying } (20) \right\}
\]
Clearly \( C_{(y, v)} \) is convex and contains zero (taking \( \alpha = 0, \ w(\cdot) \equiv 0 \)). Accordingly instead of (21) here we define
\[
u_{\alpha}(t) := \mathcal{U}(t) + \alpha \mathcal{V}(t) + \frac{1}{2} \epsilon^2 \sum_{\ell=1}^{l} \sum_{i=0}^{M} \theta_i d_{i, \ell}^r \mathcal{W}_i(t), \quad t \in I,
\]
and proceed as in the proof of Theorem 4.1. Note that in the present situation, the expansions of \( \Phi \) and \( \nu_{\alpha} \) around \( \epsilon = 0 \) contain no cross product terms such as, for example, \( \Phi_{zz}(t) y(t) \). The corresponding terms \( \Phi_0 \) and \( \Phi_2 \) are still nonpositive in here and thus the proof proceeds exactly as in Theorem 4.1.

7. CONCLUSION

In this paper, we have developed a new method for deriving second order necessary conditions for optimal control problems. The method is applicable to a wide range of optimal control problems including state or mixed state-control inequality constraints. It is also suitable for control systems with distributed parameters.

REFERENCES


